

# Linear balanceable and subcubic balanceable graphs

Pierre Aboulker\*, Marko Radovanović†, Nicolas Trotignon‡,  
Théophile Trunck§ and Kristina Vušković¶

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## Abstract

In [Structural properties and decomposition of linear balanced matrices, *Mathematical Programming*, 55:129–168, 1992], Conforti and Rao conjectured that every balanced bipartite graph contains an edge that is not the unique chord of a cycle. We prove this conjecture for balanced bipartite graphs that do not contain a cycle of length 4 (also known as linear balanced bipartite graphs), and for balanced bipartite graphs whose maximum degree is at most 3. We in fact obtain results for more general classes, namely linear balanceable and subcubic balanceable graphs. Additionally, we prove that cubic balanced graphs contain a pair of twins, a result that was conjectured by Morris, Spiga and Webb in [Balanced Cayley graphs and balanced planar graphs, *Discrete Mathematics*, 310:3228–3235, 2010].

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\*Université Paris 7, LIAFA, Case 7014, 75205 Paris Cedex 13, France. E-mail: pierre.aboulker@liafa.jussieu.fr.

†Faculty of Computer Science (RAF), Union University, Knez Mihailova 6/VI, 11000 Belgrade, Serbia. E-mail: mradovanovic@raf.edu.rs. Partially supported by Serbian Ministry of Education and Science project 174033.

‡CNRS, LIP, ENS Lyon, INRIA, Université de Lyon, 15 parvis René Descartes BP 7000 69342 Lyon cedex 07 France. E-mail: nicolas.trotignon@ens-lyon.fr.

§ENS Lyon, LIP, CNRS, INRIA, Université de Lyon, 15 parvis René Descartes BP 7000 69342 Lyon cedex 07 France. E-mail: theophile.trunck@ens-lyon.fr.

¶School of Computing, University of Leeds, Leeds LS2 9JT, UK and Faculty of Computer Science (RAF), Union University, Knez Mihailova 6/VI, 11000 Belgrade, Serbia. E-mail: k.vuskovic@leeds.ac.uk. Partially supported by EPSRC grant EP/H021426/1 and Serbian Ministry of Education and Science projects 174033 and III44006.

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# 1 Introduction

A  $0, 1$  matrix is *balanced* if for every square submatrix with two ones per row and column, the number of ones is a multiple of four. This notion was introduced by Berge [1], and later extended to  $0, \pm 1$  matrices by Truemper [16]. A  $0, \pm 1$  matrix is *balanced* if for every square submatrix with two nonzero entries per row and column, the sum of the entries is a multiple of four. These matrices have been studied extensively in literature, due to their important polyhedral properties, for a survey see [8].

Given a  $0, 1$  matrix  $A$ , the *bipartite graph representation of  $A$*  is the bipartite graph having a vertex for every row in  $A$ , a vertex for every column of  $A$ , and an edge  $ij$  joining row  $i$  to column  $j$  if and only if the entry  $a_{ij}$  of  $A$  equals 1. We say that  $G$  is *balanced* if it is the bipartite graph representation of some balanced matrix. It is easy to see that a bipartite graph  $G$  is balanced if and only if every hole of  $G$  has length  $0 \pmod{4}$ , where a *hole* is a chordless cycle of length at least 4. A *signed bipartite graph* is a bipartite graph, together with an assignment of weights  $+1, -1$  to the edges of  $G$ . A signed bipartite graph is *balanced* if the *weight* of every hole  $H$  of  $G$ , i.e. the sum of the weights of the edges of  $H$ , is  $0 \pmod{4}$ . A bipartite graph is *balanceable* if there exists a *signing* of its edges, i.e. an assignment of weights  $+1, -1$  to the edges of the graphs, such that the resulting signed bipartite graph is balanced.

The following conjecture is the last unresolved conjecture about balanced (balanceable) bipartite graphs in Cornuéjols' book [11] (it is Conjecture 6.11). Note that Conjectures 9.23, 9.28 and 9.29 from [11] have been resolved by Chudnovsky and Seymour in [4].

**Conjecture 1.1 (Conforti and Rao [10])** *Every balanced bipartite graph contains an edge that is not the unique chord of a cycle.*

In other words, every balanced bipartite graph contains an edge whose removal leaves the graph balanced. This is not true if the graph is balanceable, as shown by  $R_{10}$ , that is the graph defined by the cycle  $x_1x_2 \dots x_{10}x_1$  (of length 10) with chords  $x_i x_{i+5}$ ,  $1 \leq i \leq 5$  (see Figure 1 – in all figures in this paper a solid line denotes an edge, and a dashed one a path of length greater than 1). Graph  $R_{10}$  is cubic and balanceable (a proper signing of  $R_{10}$  is to assign weight  $+1$  to the edges of the cycle  $x_1x_2 \dots x_{10}x_1$  and  $-1$  to the chords), but not balanced ( $x_1x_2x_3x_4x_5x_6$  is a hole of length 6). Note that in  $R_{10}$  every edge is the unique chord of some cycle. Conjecture 1.1 generalises to balanceable graphs in the following way.

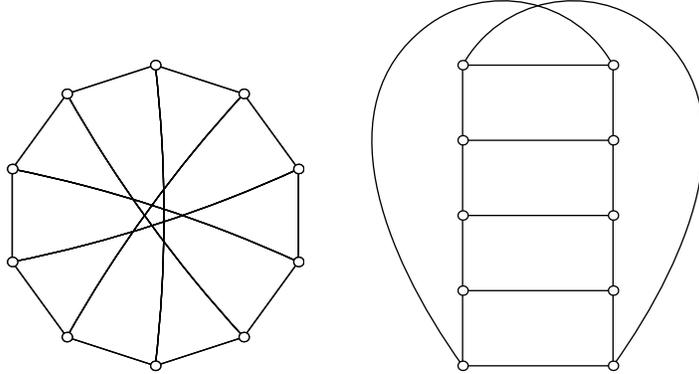


Figure 1: Two ways to draw the graph  $R_{10}$ .

**Conjecture 1.2 (Conforti, Cornuéjols and Vušković [8])** *In a balanceable bipartite graph either every edge belongs to some  $R_{10}$  or there is an edge that is not the unique chord of a cycle.*

These conjectures are known to be true for several classes of graphs. A bipartite graph is *restricted balanceable* if there exists a signing of its edges so that in the resulting signed graph every cycle (induced or not) is balanced. Clearly no edge of a restricted balanceable bipartite graph can be the unique chord of a cycle. In other words, the removal of any subset of edges from a restricted balanceable graph leaves the graph restricted balanceable. A bipartite graph is *strongly balanceable* if it is balanceable and does not contain a cycle with the unique chord. Figure 2 shows that there are cubic balanceable graphs that are not strongly balanceable. This class generalizes restricted balanceable graphs, and it clearly satisfies Conjecture 1.2. On the other hand, removing any edge from a strongly balanceable graph might not leave the graph strongly balanceable. In [9] it is shown that every strongly balanceable graph has an edge whose removal leaves the graph strongly balanceable. A bipartite graph is *totally balanced* if every hole of  $G$  is of length 4. It is shown in [12] that every totally balanced bipartite graph has a bisimplicial edge (i.e. an edge  $uv$  such that the node set  $N(u) \cup N(v)$  induces a complete bipartite graph). So clearly, the graph obtained by removing a bisimplicial edge from a totally balanced bipartite graph is also totally balanced.

A bipartite graph is *linear balanceable* if it is balanceable and does not contain a 4-hole (i.e. a hole of length 4). A graph  $G$  is *subcubic* if  $\Delta(G) \leq 3$ .

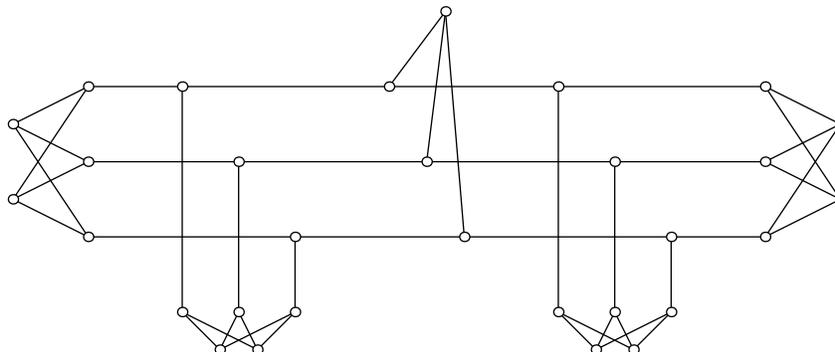


Figure 2: Cubic balanceable graph that is not strongly balanceable.

In this paper, we prove that conjectures 1.1 and 1.2 hold when restricted to linear balanceable graphs (see Corollary 4.3) and to subcubic balanceable graphs (see Corollary 5.4). For the subcubic case, our proof relies on a result conjectured by Morris, Spiga and Webb [13], stating that every cubic balanced bipartite graph contains a pair of vertices with the same neighborhood (see Corollary 5.3).

Our proofs are based on known decomposition theorems for the classes we consider, which we describe in Section 2. The decomposition theorems say that either the graph belongs to some simple subclass, that we call basic, or it has a 2-join, 6-join or star cutset. It is not straightforward to use these decomposition theorems to prove the desired results. In fact, the decomposition theorem for balanced bipartite graphs [7] has been known since the early 1990's, and still no one knows how to use it to prove the Conforti and Rao Conjecture. The key idea that makes things work for us, is the use of extreme decompositions, i.e. decompositions in which one of the blocks is basic. In Section 3 we prove that if star cutsets are excluded, then the graphs in our classes admit extreme decompositions. This is sufficient for the proof of the main result in the subcubic case in Section 5, since the induction hypothesis in this case goes through the star cutset nicely. For the linear balanceable bipartite graphs, this is not the case. Here we cannot inductively get rid of star cutsets in a straightforward manner. Furthermore, it is not true that if a (linear balanceable) graph has a star cutset, then it has a star cutset one of whose blocks of decomposition does not have a star cutset. Instead, to prove the main result for linear balanceable graphs in Section 4, we develop a new technique for finding an "extreme decomposition" with respect to star cutsets: we look for a minimally-sided double star

cutset, and show that the corresponding block of decomposition does not have a star cutset.

## Terminology

We say that a graph  $G$  *contains* a graph  $H$  if  $H$  is isomorphic to an induced subgraph of  $G$ . A graph  $G$  is  *$H$ -free* if it does not contain  $H$ . For  $x \in V(G)$ ,  $N(x)$  denotes the set of neighbors of  $x$ . For  $S \subseteq V(G)$ ,  $G[S]$  denotes the subgraph of  $G$  induced by  $S$ , and  $G \setminus S = G[V(G) \setminus S]$ . For  $S \subseteq E(G)$ ,  $G \setminus S$  denotes the graph obtained from  $G$  by deleting edges from  $S$ .

A *path*  $P$  is a sequence of distinct vertices  $p_1 p_2 \dots p_k$ ,  $k \geq 1$ , such that  $p_i p_{i+1}$  is an edge for all  $1 \leq i < k$ . Edges  $p_i p_{i+1}$ , for  $1 \leq i < k$ , are called the *edges of  $P$* . Vertices  $p_1$  and  $p_k$  are the *ends* of  $P$ . A *cycle*  $C$  is a sequence of vertices  $p_1 p_2 \dots p_k p_1$ ,  $k \geq 3$ , such that  $p_1 \dots p_k$  is a path and  $p_1 p_k$  is an edge. Edges  $p_i p_{i+1}$ , for  $1 \leq i < k$ , and edge  $p_1 p_k$  are called the *edges of  $C$* . Let  $Q$  be a path or a cycle. The vertex set of  $Q$  is denoted by  $V(Q)$ . The *length* of  $Q$  is the number of its edges. An edge  $e = uv$  is a *chord* of  $Q$  if  $u, v \in V(Q)$ , but  $uv$  is not an edge of  $Q$ . A path or a cycle  $Q$  in a graph  $G$  is *chordless* if no edge of  $G$  is a chord of  $Q$ . *Girth* of a graph is the length of its shortest cycle.

A *cut vertex* of a connected graph  $G$  is a vertex  $v$  such that  $G \setminus \{v\}$  is disconnected. A *block* of a graph is a connected subgraph that has no cut vertex and that is maximal with respect to this property. We may associate with any graph  $G$  a graph  $B(G)$  on  $\mathcal{B} \cup S$ , where  $\mathcal{B}$  is the set of blocks of  $G$  and  $S$  the set of cut vertices of  $G$ , a block  $B$  and a cut vertex  $v$  being adjacent if and only if  $B$  contains  $v$ . It is a classical result that  $B(G)$  is a tree (see [2]). The blocks that correspond to leaves of  $B(G)$  are the *end blocks* of  $G$ .

## 2 Decomposition theorems

In this section we describe known decomposition theorems for balanceable graphs. First, we state the forbidden induced subgraph characterization of balanceable graphs. Let  $G$  be a bipartite graph. Let  $u, v$  be two nonadjacent vertices of  $G$ . A *3-path configuration connecting  $u$  and  $v$* , is defined by three chordless paths  $P_1, P_2, P_3$  with ends  $u$  and  $v$ , such that the vertex set  $V(P_i) \cup V(P_j)$  induces a hole, for  $i, j \in \{1, 2, 3\}$  and  $i \neq j$ . A 3-path configuration is said to be *odd* if it connects two vertices that are on opposite sides of the bipartition. A *wheel* is defined by a hole  $H$  and a vertex  $x \notin V(H)$  having at least three neighbors in  $H$ , say  $x_1, x_2, \dots, x_n$ . If  $n$  is even

then the wheel is an *even wheel*, and otherwise it is an *odd wheel*. A 3-path configuration and an odd wheel are shown in Figure 3.

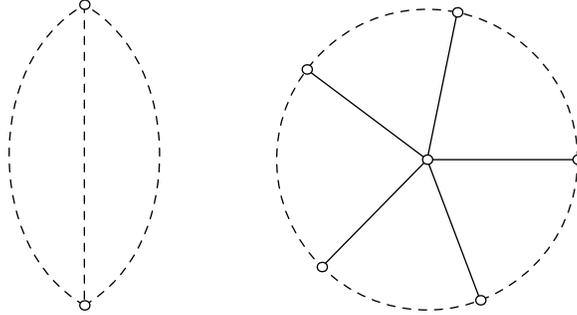


Figure 3: 3-path configuration and an odd wheel.

It is easy to see that a balanceable graph does not contain an odd 3-path configuration, nor an odd wheel. The following theorem of Truemper states that the converse is also true.

**Theorem 2.1 (Truemper [16])** *A bipartite graph is balanceable if and only if it does not contain an odd wheel nor an odd 3-path configuration.*

Now, we introduce different cutsets used in the decomposition theorems that we need.

A set  $S$  of vertices (resp. edges) of a connected graph  $G$  is a *vertex cutset* (resp. *edge cutset*) if the subgraph  $G \setminus S$  is disconnected.

### 1-join

A graph  $G$  has a *1-join* if  $V(G)$  can be partitioned into sets  $X$  and  $Y$  so that the following hold:

- $|X| \geq 2$  and  $|Y| \geq 2$ .
- There exist sets  $A$  and  $B$  such that  $\emptyset \neq A \subseteq X$  and  $\emptyset \neq B \subseteq Y$ ; there are all possible edges between  $A$  and  $B$ ; and there are no other edges between  $X$  and  $Y$ .

We say that  $(X, Y, A, B)$  is a *split* of this 1-join.

### 2-join

A graph  $G$  has a *2-join*  $(X_1, X_2)$  if  $V(G)$  can be partitioned into sets  $X_1$  and  $X_2$  so that the following hold:

- For  $i = 1, 2$ ,  $X_i$  contains disjoint nonempty sets  $A_i$  and  $B_i$ , such that every vertex of  $A_1$  is adjacent to every vertex of  $A_2$ , every vertex of  $B_1$  is adjacent to every vertex of  $B_2$ , and there are no other adjacencies between  $X_1$  and  $X_2$ .
- For  $i = 1, 2$ ,  $X_i$  contains at least one path from  $A_i$  to  $B_i$ , and if  $|A_i| = |B_i| = 1$  then  $G[X_i]$  is not a chordless path.

We say that  $(X_1, X_2, A_1, A_2, B_1, B_2)$  is a *split* of this 2-join, and the sets  $A_1, A_2, B_1, B_2$  are the *special sets* of this 2-join.

### 6-join

A graph  $G$  has a  $\theta$ -join  $(X_1, X_2)$  if  $V(G)$  can be partitioned into sets  $X_1$  and  $X_2$  so that the following hold:

- $X_1$  (resp.  $X_2$ ) contains disjoint nonempty sets  $A_1, A_3, A_5$  (resp.  $A_2, A_4, A_6$ ) such that, for every  $i \in \{1, \dots, 6\}$ , every vertex in  $A_i$  is adjacent to every vertex in  $A_{i-1} \cup A_{i+1}$  (where subscripts are taken modulo 6), and these are the only adjacencies between  $X_1$  and  $X_2$ .
- $|X_1| \geq 4$  and  $|X_2| \geq 4$ .

We say that  $(X_1, X_2, A_1, A_2, A_3, A_4, A_5, A_6)$  is a *split* of this 6-join.

### Extended star cutset

In a connected bipartite graph  $G$ ,  $(x, T, A, R)$  is an *extended star cutset* if  $T, A, R$  are disjoint subsets of  $V(G)$ ,  $x \in T$  and the following hold:

- The graph  $G \setminus (T \cup A \cup R)$  is disconnected.
- $A \cup R \subseteq N(x)$
- The vertex set  $T \cup A$  induces a complete bipartite graph (with vertex set  $T$  on one side of the bipartition and vertex set  $A$  on the other).
- If  $|T| \geq 2$ , then  $|A| \geq 2$ .

An extended star cutset such that  $T = \{x\}$  is a *star cutset*. In this paper we will denote it as  $(x, R)$ . Note that when  $|T| = 1$  and  $A \cup R = \emptyset$  then  $\{x\}$  is a cut vertex.

The following theorem is proved in [5], building on the decomposition theorem in [7]. We observe that the definition of 2-join in [7] and [5] is slightly different from the one we gave here. We define the 2-join and state the following theorem as in [8]. The statement is easily seen to be equivalent to the one in [5] by Lemma 2.5 below.

**Theorem 2.2 (Conforti, Cornuéjols, Kapoor and Vušković [5])** *A connected balanceable bipartite graph is either strongly balanceable or is  $R_{10}$ , or it has a 2-join, a 6-join or an extended star cutset.*

**Theorem 2.3 (Conforti and Rao [9])** *A strongly balanceable bipartite graph is either restricted balanceable or has a 1-join.*

A bipartite graph is *basic* if it admits a bipartition such that all the vertices in one side of the bipartition have degree at most 2.

**Theorem 2.4 (Yannakakis [18])** *A restricted balanceable bipartite graph is either basic or has a cut vertex or a 2-join whose special sets are all of size 1.*

The following lemma is proved in [15] (Lemma 3.2) and a special case of it is proved in [7] (Lemma 2.4).

**Lemma 2.5** *Let  $G$  be a graph that has no star cutset, and let  $(X_1, X_2, A_1, A_2, B_1, B_2)$  be a split of a 2-join of  $G$ . Then for  $i = 1, 2$ , the following hold:*

- (i) *Every component of  $G[X_i]$  meets both  $A_i$  and  $B_i$ .*
- (ii) *Every  $u \in X_i$  has a neighbor in  $X_i$ .*
- (iii) *Every vertex of  $A_i$  has a non-neighbor in  $B_i$ .*
- (iv) *Every vertex of  $B_i$  has a non-neighbor in  $A_i$ .*
- (v)  $|X_i| \geq 4$ .

**Lemma 2.6** *Let  $G$  be a bipartite graph that has no star cutset. If  $G$  has a 1-join, then  $G$  is a 4-hole.*

PROOF — Let  $(X, Y, A, B)$  be a split of a 1-join of  $G$ . If  $Y \setminus B \neq \emptyset$ , then a vertex from  $A$  and set  $B$  form a star cutset, a contradiction. So  $Y = B$ , and by symmetry  $X = A$ . If  $|A| \geq 3$ , then a vertex from  $A$  and set  $B$  form a star cutset, a contradiction. So, by symmetry,  $|A| = |B| = 2$ , and therefore  $G$  is a 4-hole.  $\square$

From Theorems 2.2, 2.3 and 2.4, Lemmas 2.5 and 2.6, and the fact that in 4-hole-free graphs and in graphs with  $\Delta(G) \leq 3$  an extended star cutset reduces to a star cutset, we get the following decomposition theorem that we will use in this paper.

**Theorem 2.7** *Let  $G$  be a connected balanceable graph.*

- *If  $G$  is 4-hole-free, then  $G$  is basic, or has a 2-join, a 6-join or a star cutset.*
- *If  $\Delta(G) \leq 3$ , then  $G$  is basic or is  $R_{10}$ , or has a 2-join, a 6-join or a star cutset.*

We observe that a balanceable bipartite graph  $G$  with  $\Delta(G) \leq 3$  is actually *regular*, as we explain now. A matrix is *totally unimodular* if every square submatrix has determinant equal to 0, +1 or  $-1$ . A 0, 1 matrix is *regular* if its nonzero entries can be signed +1 or  $-1$  so that the resulting matrix is totally unimodular. A 0, 1 matrix  $A$  can be thought of as a vertex-vertex incidence matrix of a bipartite graph, which we denote with  $G(A)$ . We say that a bipartite graph  $G$  is *regular* if  $G = G(A)$  for some regular 0, 1 matrix  $A$ . A graph is *eulerian* if all its vertices have even degree. By a theorem of Camion [3], a bipartite graph is regular if and only if there exists a signing of its edges with +1 or  $-1$  so that the weight of every induced eulerian subgraph is a multiple of 4. It now clearly follows that for a bipartite graph  $G$  with  $\Delta(G) \leq 3$ :  $G$  is balanceable if and only if  $G$  is regular.

It is natural to ask why we use Theorem 2.7 in our proof of Conforti and Rao Conjecture in the subcubic case, instead of Seymour's decomposition theorem for regular bipartite graphs [14]. The answer is that by using Theorem 2.7 we have only to check whether three cutsets (2-join, 6-join and star cutset) go through our induction hypothesis, whereas if we used the decomposition theorem in [14] we would have to check five cutsets (1-join, 2-join, 6-join, N-join and M-join, for an explanation see [17]). Furthermore, 2-joins and 6-joins in graphs with no star cutset have special properties (given in Section 3) which are very useful for pushing the induction hypothesis through them.

### 3 Graphs with no star cutset

The following properties of graphs with no star cutsets will be essential in our proofs.

Let  $(X_1, X_2, A_1, A_2, B_1, B_2)$  be a split of a 2-join of a graph  $G$ . The *blocks of decomposition* of  $G$  by this 2-join are graphs  $G_1$  and  $G_2$  defined as follows. To obtain  $G_i$ , for  $i = 1, 2$ , we start from  $G[X_i]$ , and first add a vertex  $a_{3-i}$ , adjacent to all the vertices in  $A_i$  and no other vertex of  $X_i$ , and a vertex  $b_{3-i}$  adjacent to all the vertices in  $B_i$  and no other vertex of  $X_i$ .

For  $i = 1, 2$ , let  $Q_{3-i}$  be a path in  $G[X_{3-i}]$  with smallest number of edges connecting a vertex in  $A_{3-i}$  to a vertex in  $B_{3-i}$ . For  $i = 1, 2$ , add to  $G_i$  a *marker path*  $M_{3-i}$  connecting  $a_{3-i}$  and  $b_{3-i}$  with length  $|E(M_{3-i})| \in \{4, 5\}$  having the same parity as  $Q_{3-i}$ .

The following lemma is proved in [6]. (Note that it is not stated like here but the proof of Theorem 4.6 in [6] shows precisely what we need).

**Lemma 3.1** *Let  $G$  be a bipartite graph with no star cutset. Let  $(X_1, X_2)$  be a 2-join of  $G$ , and let  $G_1$  and  $G_2$  be the corresponding blocks of decomposition. Then the following hold:*

- (i) *If  $G$  is balanceable then  $G_1$  and  $G_2$  are balanceable.*
- (ii)  *$G_1$  and  $G_2$  have no star cutset.*
- (iii) *If  $G$  has no 6-join then  $G_1$  and  $G_2$  have no 6-join.*

Let  $(X_1, X_2, A_1, \dots, A_6)$  be a split of a 6-join of a graph  $G$ . The *blocks of decomposition* of  $G$  by this 6-join are graphs  $G_1$  and  $G_2$  defined as follows. For  $i = 1, \dots, 6$  let  $a_i$  be any vertex of  $A_i$ . Then  $G_1 = G[X_1 \cup \{a_2, a_4, a_6\}]$  and  $G_2 = G[X_2 \cup \{a_1, a_3, a_5\}]$ . Nodes  $a_2, a_4, a_6$  (resp.  $a_1, a_3, a_5$ ) are called the *marker nodes* of  $G_1$  (resp.  $G_2$ ).

**Lemma 3.2** *Let  $G$  be a bipartite graph with no star cutset. Let  $(X_1, X_2, A_1, \dots, A_6)$  be a split of a 6-join of  $G$ , and  $G_1$  and  $G_2$  the corresponding blocks of decomposition. Then the following hold:*

- (i)  $X_1 \setminus (A_1 \cup A_3 \cup A_5) \neq \emptyset$  and  $X_2 \setminus (A_2 \cup A_4 \cup A_6) \neq \emptyset$ .
- (ii) *If  $C$  is a connected component of  $G[X_1 \setminus (A_1 \cup A_3 \cup A_5)]$  (resp.  $G[X_2 \setminus (A_2 \cup A_4 \cup A_6)]$ ), then a node of  $A_i$ , for every  $i = 1, 3, 5$  (resp.  $i = 2, 4, 6$ ) has a neighbor in  $C$ .*
- (iii) *If  $G$  is 4-hole-free or  $\Delta(G) \leq 3$ , then  $|A_i| = 1$  for every  $i \in \{1, \dots, 6\}$ , and in particular every node of  $\cup_{i=1}^6 A_i$  is of degree at least 3 in  $G$ .*
- (iv) *If  $G$  is balanceable then so are  $G_1$  and  $G_2$ .*
- (v) *If  $G$  is 4-hole-free, then  $G_1$  and  $G_2$  do not have star cutsets.*

PROOF — Suppose that  $X_1 \setminus (A_1 \cup A_3 \cup A_5) = \emptyset$ . Then w.l.o.g.  $|A_1| \geq 2$ , and hence for a node  $a_1 \in A_1$ ,  $\{a_1\} \cup A_2 \cup A_6$  is a star cutset of  $G$ , a contradiction. Therefore (i) holds.

Let  $C$  be a connected component of  $G[X_1 \setminus (A_1 \cup A_3 \cup A_5)]$  and suppose that no node of  $A_1$  has a neighbour in  $C$ . Then for a node  $a_4 \in A_4$ ,  $\{a_4\} \cup A_3 \cup A_5$  is a star cutset of  $G$  separating  $C$  from the rest, a contradiction. Therefore by symmetry, (ii) holds.

If  $G$  is 4-hole-free then clearly  $|A_i| = 1$  for every  $i \in \{1, \dots, 6\}$ , and if  $\Delta(G) \leq 3$  then the same holds by (i) and (ii), therefore, (iii) holds.

Since  $G_1$  and  $G_2$  are induced subgraphs of  $G$ , (iv) holds.

To prove (v) assume  $G$  is 4-hole-free and w.l.o.g.  $G_1$  has a star cutset  $(x, R)$ . Let  $a_2, a_4, a_6$  be the marker nodes of  $G_1$ . By (ii),  $x \notin \{a_2, a_4, a_6\}$ . If  $x \in A_1$  then  $(x, R \cup A_2 \cup A_6)$  is a star cutset of  $G$ , a contradiction. Therefore by symmetry,  $x \in X_1 \setminus (A_1 \cup A_3 \cup A_5)$ . Since  $G$  is 4-hole-free  $R$  may contain nodes from at most one of the sets  $A_1, A_3, A_5$ , and hence  $a_2, a_4, a_6$  are all contained in the same connected component of  $G_1 \setminus (\{x\} \cup R)$ . It follows that  $(x, R)$  is also a star cutset of  $G$ , a contradiction. Therefore (v) holds.  $\square$

We observe that property (v) above is not true in general for balanceable graphs. On the other hand, it is true for subcubic balanceable graphs. Since we will use a different technique to prove the main result for subcubic balanceable graphs, than the one we will use for linear balanceable graphs, we will not need this result.

A 2-join  $(X_1, X_2)$  of  $G$  is a *minimally-sided 2-join* if for some  $i \in \{1, 2\}$  the following holds: for every 2-join  $(X'_1, X'_2)$  of  $G$ , neither  $X'_1 \subsetneq X_i$  nor  $X'_2 \subsetneq X_i$ . In this case  $X_i$  is a *minimal side* of this minimally-sided 2-join.

**Lemma 3.3 (Trotignon and Vušković [15])** *Let  $G$  be a graph with no star cutset. Let  $(X_1, X_2, A_1, A_2, B_1, B_2)$  be a split of a minimally-sided 2-join of  $G$  with  $X_1$  being a minimal side, and let  $G_1$  and  $G_2$  be the corresponding blocks of decomposition. Then the following hold:*

(i)  $|A_1| \geq 2$ ,  $|B_1| \geq 2$ , and in particular all the vertices of  $A_2 \cup B_2$  are of degree at least 3.

(ii) If  $G_1$  and  $G_2$  do not have star cutsets, then  $G_1$  has no 2-join.

A partition  $(X_1, X_2)$  of  $V(G)$  is a  $\{2, 6\}$ -join if it is a 2-join or a 6-join of  $G$ . It is a *minimally-sided  $\{2, 6\}$ -join* if for some  $i \in \{1, 2\}$  the following holds: for every  $\{2, 6\}$ -join  $(X'_1, X'_2)$  of  $G$ , neither  $X'_1 \subsetneq X_i$  nor  $X'_2 \subsetneq X_i$ . In this case  $X_i$  is a *minimal side* of this minimally-sided  $\{2, 6\}$ -join.

**Lemma 3.4** *Let  $G$  be a 4-hole-free bipartite graph. Let  $(X_1, X_2)$  be a minimally-sided  $\{2, 6\}$ -join of  $G$ , with  $X_1$  being a minimal side. If  $G$  has no star cutset, then the block of decomposition  $G_1$  has no  $\{2, 6\}$ -join.*

PROOF — Assume not and let  $(X'_1, X'_2)$  be a  $\{2, 6\}$ -join of  $G_1$ . We now consider the following cases.

**Case 1:**  $(X_1, X_2)$  is a 2-join of  $G$ .

By Lemmas 3.1 (ii) and 3.3 (ii),  $(X'_1, X'_2)$  is a 6-join of  $G$ , say with split  $(X'_1, X'_2, A'_1, \dots, A'_6)$ . Let  $P_2$  be the marker path of  $G_1$ . By Lemma 3.2 (iii), we may assume w.l.o.g. that  $V(P_2) \subseteq X'_2$ . If  $V(P_2) \subseteq X'_2 \setminus (A'_2 \cup A'_4 \cup A'_6)$  then clearly  $(X'_1, X'_2 \setminus V(P_2) \cup X_2)$  is a 6-join of  $G$  that contradicts the choice of  $(X_1, X_2)$ . So  $V(P_2) \cap (A'_2 \cup A'_4 \cup A'_6) \neq \emptyset$ . By Lemma 3.2 (ii), we may assume w.l.o.g. that  $V(P_2) \cap (A'_4 \cup A'_6) = \emptyset$ . But then  $(X'_1, X'_2 \setminus V(P_2) \cup X_2, A'_1, A_2, A'_3, A'_4, A'_5, A'_6)$  is a split of a 6-join of  $G$  that contradicts the choice of  $(X_1, X_2)$ .

**Case 2:**  $(X_1, X_2)$  is a 6-join of  $G$ .

Let  $(X_1, X_2, A_1, \dots, A_6)$  be the split of this 6-join, and let  $a_2, a_4, a_6$  be the marker nodes of  $G_1$ . We now consider the following two cases.

**Case 2.1:**  $(X'_1, X'_2)$  is a 6-join of  $G_1$ .

Let  $(X'_1, X'_2, A'_1, \dots, A'_6)$  be the split of this 6-join. By Lemma 3.2 (iii) we may assume w.l.o.g. that  $\{a_2, a_4, a_6\} \subseteq X'_2 \setminus (A'_2 \cup A'_4 \cup A'_6)$ . But then  $(X'_1, X'_2 \cup X_2)$  is a 6-join of  $G$  that contradicts the choice of  $(X_1, X_2)$ .

**Case 2.2:**  $(X'_1, X'_2)$  is a 2-join of  $G_1$ .

Let  $(X'_1, X'_2, A'_1, A'_2, B'_1, B'_2)$  be the split of this 2-join. By Lemma 3.2 (iii), let  $A_1 = \{a_1\}$ ,  $A_3 = \{a_3\}$  and  $A_5 = \{a_5\}$ , and let  $H$  be the 6-hole induced by  $\{a_1, \dots, a_6\}$ . First suppose that both  $X'_1 \setminus (A'_1 \cup B'_1)$  and  $X'_2 \setminus (A'_2 \cup B'_2)$  contain a node of  $H$ . Then w.l.o.g. we may assume that  $a_2 \in X'_2 \setminus (A'_2 \cup B'_2)$ ,  $a_4 \in B'_1$  and  $a_6 \in A'_1$ . Since nodes  $a_2, a_4$  and  $a_6$  are all of degree 2 in  $G_1$ , it follows that  $A'_2 = \{a_1\}$  and  $B'_2 = \{a_3\}$ , and hence by Lemma 3.2 (iii)  $(a_2, \{a_1, a_3\})$  is a star cutset of  $G$ , a contradiction.

So we may assume w.l.o.g. that  $(X'_2 \setminus (A'_2 \cup B'_2)) \cap V(H) = \emptyset$ . By Lemma 2.5 (ii) and since  $a_2, a_4, a_6$  are all of degree 2 in  $G_1$ , it follows that in fact w.l.o.g. we may assume that  $V(H) \cap X'_2 \subseteq A'_2$ . By Lemma 2.5 (ii), every node of  $A'_2$  has a neighbour in  $X'_2$ , and hence (since  $a_2, a_4, a_6$  are all of degree 2 in  $G_1$ )  $\{a_2, a_4, a_6\} \subseteq X'_1$ . But then  $(X'_1 \cup X_2, X'_2)$  is a 2-join of  $G$  that contradicts the choice of  $(X_1, X_2)$ .  $\square$

## 4 Linear balanceable graphs

A *double star cutset* of a connected graph  $G$  is a set  $S$  of vertices such that  $G \setminus S$  is disconnected and  $S$  contains two adjacent vertices  $u$  and  $v$  such that every vertex of  $S$  is adjacent to at least one of  $u$  or  $v$ . Note that a star cutset is either a double star cutset or a cut vertex. If  $U = (N(u) \cap S) \setminus \{v\}$  and  $V = (N(v) \cap S) \setminus \{u\}$ , then this double star cutset is denoted by  $(u, v, U, V)$ . Note that if  $G$  is a 4-hole-free bipartite graph,  $U \cup V$  induce a stable set and  $U \cap V = \emptyset$ .

Let  $C_i$ , for  $i = 1, 2$ , be a partition of the vertex set  $V(G \setminus S)$ , such that there are no edges between vertices of  $C_1$  and  $C_2$ . Then  $G_i = G[S \cup V(C_i)]$ ,  $i = 1, 2$ , are *blocks of decomposition* with respect to this double star cutset.

A double star cutset of a 2-connected graph  $G$  with blocks of decompositions  $G_1$  and  $G_2$  is a *minimally-sided double star cutset* if for some  $i \in \{1, 2\}$  the following holds: for every double star cutset of  $G$  with blocks of decompositions  $G'_1$  and  $G'_2$  neither  $V(G'_1) \subsetneq V(G_i)$  nor  $V(G'_2) \subsetneq V(G_i)$ . In this case  $G_i$  is a *minimal side* of this minimally-sided double star cutset.

**Lemma 4.1** *Let  $G$  be a 2-connected 4-hole-free bipartite graph that has a star cutset. Let  $G_i$ , for some  $i \in \{1, 2\}$  be a minimal side of a minimally-sided double star cutset of  $G$ . Then  $G_i$  does not have a star cutset.*

PROOF — Let  $(u, v, U, V)$  be a minimally-sided double star cutset, let  $G_1$  be its minimal side, and let  $S = \{u, v\} \cup U \cup V$ . Observe that every vertex of  $U \cup V$  has a neighbor in  $G_1 \setminus S$ . In particular,  $G_1$  is 2-connected. Let us assume by way of contradiction that  $(x, R)$  is a star cutset of  $G_1$ . Since  $G_1$  is 2-connected,  $R \neq \emptyset$ .

**Case 1:**  $x \notin S$ .

Since  $G$  is 4-hole-free and bipartite,  $x$  has at most one neighbor in  $S$ . If  $R \cap \{u, v\} = \emptyset$ , then vertices of  $S \setminus R$  are in the same connected component of  $G_1 \setminus (\{x\} \cup R)$ , and therefore  $(x, y, R \setminus \{y\}, \emptyset)$ , for a vertex  $y \in R$ , is a double star cutset of  $G$  that contradicts the minimality of  $G_1$ . So w.l.o.g.  $u \in R$ . Let  $C$  be a connected component of  $G_1 \setminus (\{x\} \cup R)$  that does not contain a node of  $\{v\} \cup V$ . If  $V(C) \setminus U \neq \emptyset$ , then  $(x, u, R \setminus \{u\}, U)$  is a double star cutset of  $G$  that contradicts the minimality of  $G_1$ . So  $V(C) \setminus U = \emptyset$ . But then some vertex  $u' \in U$  is of degree 1 in  $G_1$  (since  $G_1$  is 4-hole-free and bipartite), contradicting the fact that  $G_1$  is 2-connected.

**Case 2:**  $x \in S$ .

First, let us assume that  $x \in \{u, v\}$ , say  $x = u$ . Since  $G$  is 4-hole-free and bipartite, every connected component of  $G_1 \setminus (\{x\} \cup R)$  that contains a vertex from  $U$  or a vertex from  $V$  contains a vertex from  $G_1 \setminus S$ . Therefore,  $(x, v, (U \cup R) \setminus \{v\}, V)$  is a double star cutset of  $G$  that contradicts the minimality of  $G_1$ . So,  $x \in U \cup V$ , and w.l.o.g. we may assume that  $x \in U$ . Then the nodes of  $\{v\} \cup V$  are all contained in the same connected component of  $G_1 \setminus (\{x\} \cup R)$ . Again, since  $G$  is 4-hole-free and bipartite, every connected component of  $G_1 \setminus (\{x\} \cup R)$  that contains a vertex from  $U$  contains a vertex from  $G_1 \setminus S$ . Therefore,  $(x, u, R \setminus \{u\}, U \setminus \{x\})$  is a double star cutset of  $G$  that contradicts the minimality of  $G_1$ .  $\square$

Our main result about linear balanceable graphs is the following.

**Theorem 4.2** *If  $G$  is a linear balanceable graph on at least two vertices, then  $G$  contains at least two vertices of degree at most 2.*

PROOF — We prove the theorem by induction on  $|V(G)|$ . If  $|V(G)| = 2$ , then the theorem trivially holds. So, let  $G$  be a linear balanceable graph such that  $|V(G)| > 2$ . We may assume that  $G$  is connected, else we are done by induction.

Let  $u$  be a cut vertex of  $G$ , and let  $\{C_1, C_2\}$  be a partition of  $V(G) \setminus \{u\}$ , such that there are no edges between vertices of  $C_1$  and  $C_2$ . Then, by induction applied to graphs  $G[C_i \cup \{u\}]$  for  $i = 1, 2$ , there is a vertex  $c_i \in C_i \setminus \{u\}$ , for  $i = 1, 2$ , that is of degree at most 2 in  $G[C_i \cup \{u\}]$ . But then  $c_1$  and  $c_2$  are also of degree at most 2 in  $G$ . So, we may assume that  $G$  is 2-connected.

Now suppose that  $G$  admits a star cutset. By Lemma 4.1, there is a double star cutset  $(u, v, U, V)$  of  $G$ , such that a block of decomposition w.r.t. this cutset, say  $G'$ , has no star cutset. Let  $S = \{u, v\} \cup U \cup V$  and note that all vertices from  $U$  and  $V$  have a neighbor in  $G' \setminus S$ . By Theorem 2.7  $G'$  is basic or has a  $\{2, 6\}$ -join.

**Case 1:**  $G'$  is basic.

Let  $(X, Y)$  be a bipartition of  $G'$  such that all vertices of  $Y$  are of degree 2. Vertices  $u$  and  $v$  are adjacent, so we may assume w.l.o.g. that  $\{v\} \cup U \subseteq Y$  and  $\{u\} \cup V \subseteq X$ . In particular,  $|V| \leq 1$ .

Suppose  $V = \{v'\}$ . All the neighbors of  $v'$  in  $G' \setminus S$  are of degree 2 in  $G'$  and in  $G$ , so we may assume that  $v'$  has a unique neighbor  $w$  in  $G' \setminus S$ . Let  $w'$  be the unique neighbor of  $w$  in  $G' \setminus v'$ . Since  $G'$  is 4-hole-free and bipartite,  $w' \in V(G') \setminus S$ . If  $w'$  is of degree 2 in  $G'$  (and hence in  $G$ ), then  $w'$  and  $w$  are the desired two vertices. So we may assume that  $w'$  has at

least three neighbours in  $G'$ . But then, since  $G'$  is 4-hole-free and bipartite,  $w'$  must have a neighbour  $w'' \in V(G') \setminus (S \cup \{w\})$ , and hence  $w$  and  $w''$  are the desired two vertices.

Now suppose that  $V = \emptyset$  and let  $v'$  be the neighbor of  $v$  in  $V(G') \setminus S$ . Since  $G$  is 4-hole-free and bipartite,  $v'$  has no neighbors in  $U \cup \{u\}$ . So, either  $\deg_{G'}(v') \geq 3$ , in which case  $v'$  has at least two neighbors in  $V(G') \setminus S$  of degree 2 in  $G'$ , and hence in  $G$ , or  $\deg_{G'}(v') = 2$ , in which case  $v'$  and the neighbor of  $v'$  in  $V(G') \setminus S$  are both of degree 2 in  $G'$ , and hence in  $G$ . Therefore  $G$  has at least two vertices of degree 2.

**Case 2:**  $G'$  has a  $\{2, 6\}$ -join.

Let  $(X'_1, X'_2)$  be a  $\{2, 6\}$ -join of  $G'$ . W.l.o.g. we may assume that  $|X'_1 \cap \{u, v\}| \leq 1$ . Let  $(X_1, X_2)$  be a minimally-sided  $\{2, 6\}$ -join of  $G'$  such that  $X_1 \subseteq X'_1$ , and let  $G_1$  be the corresponding block of decomposition. Clearly  $G_1$  is 4-hole-free and  $|X_1 \cap \{u, v\}| \leq 1$ . By Lemmas 3.1 and 3.2,  $G_1$  is linear balanceable and has no star cutset. By Lemma 3.4,  $G_1$  has no  $\{2, 6\}$ -join, and hence by Theorem 2.7,  $G_1$  is basic. We now consider the following two cases.

**Case 2.1:**  $(X_1, X_2)$  is a 6-join of  $G'$ .

Let  $(X_1, X_2, A_1, \dots, A_6)$  be the split of this 6-join. By Lemma 3.2,  $A_1 = \{a_1\}$ ,  $A_3 = \{a_3\}$ ,  $A_5 = \{a_5\}$ , and all these nodes are of degree at least 3 in  $G_1$ . Since  $G_1$  is 4-hole-free, nodes  $a_1, a_3, a_5$  do not have common neighbours in  $X_1$ . Since  $|X_1 \cap \{u, v\}| \leq 1$ , we may assume w.l.o.g. that  $(X_1 \setminus \{a_1\}) \cap \{u, v\} = \emptyset$ . Let  $a'_3$  (resp.  $a'_5$ ) be a neighbour of  $a_3$  (resp.  $a_5$ ) in  $X_1$ . Then  $a'_3 \neq a'_5$  and  $\{a'_3, a'_5\} \cap S = \emptyset$ . Since  $G_1$  is basic,  $a'_3$  and  $a'_5$  are of degree 2 in  $G_1$ , and hence in  $G'$ . Since  $\{a'_3, a'_5\} \cap S = \emptyset$ , they are also of degree 2 in  $G$ .

**Case 2.2:**  $(X_1, X_2)$  is a 2-join of  $G'$ .

Let  $(X_1, X_2, A_1, A_2, B_1, B_2)$  be the split of this 2-join, and let  $P_2$  be the marker path of  $G_1$ . By Lemma 3.3,  $|A_1| \geq 2$ ,  $|B_1| \geq 2$  and the ends of  $P_2$  are of degree at least 3 in  $G_1$ . Since  $G_1$  is basic, it follows that the nodes of  $A_1 \cup B_1$  are all of degree 2 in  $G_1$ , and on the same side of bipartition of  $G_1$ , and hence of  $G'$  as well. In particular, it is not possible that both  $u$  and  $v$  are in  $A_2 \cup B_2$ . Since  $G'$  is 4-hole-free and bipartite, it follows that  $|A_2| = |B_2| = 1$ , and hence the nodes of  $A_1 \cup B_1$  are of degree 2 in  $G'$ . Since  $|X_1 \cap \{u, v\}| \leq 1$ , w.l.o.g.  $B_1 \cap S = \emptyset$ , and hence the nodes of  $B_1$  are also of degree 2 in  $G$ .

So, we may assume that  $G$  does not admit a star cutset. Thus, by Theorem 2.7  $G$  is basic or has a  $\{2, 6\}$ -join. So the theorem holds by the same proof as in Cases 1 and 2 above.  $\square$

**Corollary 4.3** *Let  $G$  be a linear balanceable graph that has at least one edge. Then there is an edge of  $G$  that is not the unique chord of a cycle.*

PROOF — Follows immediately from Theorem 4.2 since an edge incident to a degree 2 vertex cannot be the unique chord of a cycle.  $\square$

## 5 Subcubic balanceable graphs

A *branch vertex* is a vertex of degree at least 3. A *branch* is a path connecting two branch vertices and containing no other branch vertices. Two branches are *non incident* if the sets of ends of the corresponding paths are disjoint. Note that a 2-connected graph that is not a cycle is edgewise partitioned into its branches. A pair of vertices  $(u, v)$  of  $G$  is a *pair of twins* in  $G$  if  $N(u) = N(v)$  and  $|N(u)| \geq 3$ . Note that a cubic bipartite graph has a pair of twins if and only if it contains a  $K_{2,3}$  as a subgraph. Note that  $R_{10}$  does not have a pair of twins.

Our main result on subcubic balanceable graphs is the following theorem.

**Theorem 5.1** *Let  $G$  be a 2-connected balanceable bipartite graph with  $\Delta(G) \leq 3$ . If  $G$  is not equal to  $R_{10}$  and has at least three branch vertices, then one of the following holds:*

- (i)  $G$  has two vertices of degree 2 that are in non-incident branches.
- (ii)  $G$  has a pair of twins and a vertex of degree 2.
- (iii)  $G$  has two disjoint pairs of twins.

In the previous theorem, if  $G$  has at least three branch vertices, then it has in fact at least four branch vertices (because 2-connected graphs have no vertex of degree 1).

The following lemma settles the case in which  $G$  does not admit a star cutset nor a 6-join. We treat this case separately because it does not need induction.

**Lemma 5.2** *Let  $G$  be a 2-connected balanceable bipartite graph with  $\Delta(G) \leq 3$ , that is not equal to  $R_{10}$  and has at least three branch vertices. If  $G$  does not have a star cutset nor a 6-join, then  $G$  has two vertices of degree 2 that are in non incident branches.*

PROOF — By Theorem 2.7,  $G$  is either basic or has a 2-join, so we consider the following two cases. Note that every vertex of  $G$  is of degree at least 2.

**Case 1:**  $G$  is basic.

Since  $G$  is basic, no two branch vertices are adjacent, and hence every branch of  $G$  contains a vertex of degree 2. Let  $a, b, c$  be distinct vertices of degree 3, such that there is a branch from  $a$  to  $b$ . There are three branches in  $G$  with end  $c$ . If one of the other ends of these branches is not  $a$  or  $b$ , the proof is complete. So we may assume w.l.o.g. that we have two branches between  $a$  and  $c$  and one branch between  $b$  and  $c$ . But then there is a branch from  $b$  with an end not in  $\{a, c\}$ , and hence the result follows.

**Case 2:**  $G$  has a 2-join.

Let  $(X_1, X_2, A_1, A_2, B_1, B_2)$  be a split of a minimally-sided 2-join of  $G$  with  $X_1$  being a minimal side. Let  $G_1$  be the corresponding block of decomposition. By Lemma 3.1,  $G_1$  is balanceable and it does not have a star cutset nor a 6-join. By Lemmas 3.3 and 2.5,  $G_1$  has no 2-join,  $|A_1| = |B_1| = 2$ , and all vertices of  $A_2 \cup B_2$  are of degree 3. So by Theorem 2.7  $G_1$  is basic.

**Claim:**  $X_1 \setminus (A_1 \cup B_1)$  contains a vertex of degree 2.

*Proof of Claim:* Assume not. Let  $(X, Y)$  be a bipartition of  $G_1$  such that all vertices of  $X$  are of degree 2. Let  $a_2, \dots, b_2$  be the marker path of  $G_1$ , with  $a_2$  complete to  $A_1$  and  $b_2$  complete to  $B_1$ . Then  $a_2$  and  $b_2$  are in  $Y$  and hence  $A_1 \cup B_1 \subseteq X$ . In particular, there are no edges in  $G[A_1 \cup B_1]$ . So by Lemma 2.5,  $X_1 \setminus (A_1 \cup B_1)$  is not empty. By our assumption  $X_1 \setminus (A_1 \cup B_1) \subseteq Y$ . So for every  $u \in X_1 \setminus (A_1 \cup B_1)$ ,  $N(u) \subseteq A_1 \cup B_1$ . But then since  $|A_1 \cup B_1| = 4$ ,  $|N(u)| \leq 3$  and the fact that each vertex of  $A_1 \cup B_1$  is of degree 2 in  $G_1$ , we have a contradiction. This completes the proof of the claim.

By the claim let  $c_1 \in X_1 \setminus (A_1 \cup B_1)$  be of degree 2 (in  $G_1$ , and hence in  $G$  as well). Let  $(X'_1, X'_2, A'_1, A'_2, B'_1, B'_2)$  be a split of a minimally-sided 2-join of  $G$  with  $X'_2$  being a minimal side and  $X'_2 \subseteq X_2$ . Then, as before,  $|A'_2| = |B'_2| = 2$ , and hence all the vertices of  $A'_1 \cup B'_1$  are of degree 3. By the claim, there is a vertex  $c_2 \in X'_2 \setminus (A'_2 \cup B'_2)$  that is of degree 2 in  $G$ .

Since  $|A_1| = |B_1| = |A'_2| = |B'_2| = 2$ , we see that no branch of  $G$  may overlap the three following sets:  $A_1 \cup B_1$ ,  $X_1 \setminus (A_1 \cup B_1)$  and  $A'_2 \cup B'_2$  (resp.  $A'_2 \cup B'_2$ ,  $X'_2 \setminus (A'_2 \cup B'_2)$  and  $A_1 \cup B_1$ ). It follows that  $c_1$  and  $c_2$  are in non-incident branches.  $\square$

*Proof of Theorem 5.1:* We proceed by induction on  $|V(G)|$ . If  $|V(G)| = 1$ , then the theorem is vacuously true. By Theorem 2.7 and Lemma 5.2, we

may assume that  $G$  has a star cutset or a 6-join.

**Proof when  $G$  has a star cutset.**

Let  $(x, R)$  be a star cutset of  $G$  such that  $|R|$  is minimum. Since  $G$  is 2-connected,  $|R| \geq 1$ , and by the choice of  $(x, R)$  and since  $G$  is cubic, every vertex of  $R$  has neighbors in every connected component of  $G \setminus (\{x\} \cup R)$ , every vertex of  $R$  is of degree 3 and  $G \setminus (\{x\} \cup R)$  has exactly two connected components, say  $C_1$  and  $C_2$ . Let  $G_i$  be the block of decomposition w.r.t. this cutset that contains  $C_i$ , for  $i = 1, 2$ . Note that every vertex of  $R$  is of degree 2 in  $G_i$ .

**Claim:** *If  $x$  is of degree 2 in  $G_i$ , for some  $i \in \{1, 2\}$ , then  $C_i$  contains a vertex  $u$  of degree 2, or a pair of twins. Furthermore, if  $G_i$  has at least two branch vertices, then  $u$  can be chosen so that  $x$  and  $u$  are not in the same branch of  $G_i$ .*

*Proof of Claim:* If  $G_i$  has no branch vertices, then  $C_i$  contains a vertex of degree 2. If  $G_i$  has exactly two branch vertices, both are in  $C_i$  and there is a branch of  $G_i$  with vertices in  $C_i$  of length at least 2, and therefore there is a vertex of degree 2 in  $C_i$  that is not in the same branch as  $x$ . If  $G_i$  has at least 3 branch vertices, then, by the induction hypothesis,  $C_i$  contains a vertex of degree 2 that is not in the same branch as  $x$ , or  $C_i$  contains a pair of twins. This completes the proof of Claim.

We now consider the following cases.

**Case 1:**  $|R| = 1$ .

Note that since  $G$  is 2-connected,  $x$  has a neighbor in both  $C_1$  and  $C_2$ , and in particular,  $x$  is of degree 2 in both  $G_1$  and  $G_2$ . Since  $G$  has at least three branch vertices, at least one of  $G_1$  or  $G_2$  has at least two branch vertices, so, by Claim applied for  $i = 1$  and  $i = 2$ ,  $G$  satisfies the theorem.

**Case 2:**  $|R| = 2$ .

Let  $R = \{y_1, y_2\}$ . If  $\deg(x) = 2$ , then at least one of  $G_1$  or  $G_2$  has at least two branch vertices, and hence by Claim,  $G$  satisfies the theorem, a contradiction. So  $\deg(x) = 3$ , and w.l.o.g.  $x$  has a neighbour in  $C_1$  and does not in  $C_2$ . If  $G_1$  has exactly two branch vertices and they are adjacent, then for a shortest path  $P$  from  $y_1$  to  $y_2$  in  $G_2 \setminus \{x\}$ , set  $V(G_1) \cup V(P)$  induces an odd wheel with centre  $x$ , contradicting Theorem 2.1. So, if  $G_1$  has exactly two branch vertices, then there is a vertex of degree 2 in  $G_1$  in a branch that does not contain  $y_1$  nor  $y_2$ , and therefore, by Claim,  $G$  satisfies the theorem, a contradiction. So  $G_1$  must have at least three branch vertices, and hence by the choice of  $G$ ,  $G_1$  has a pair of twins or a vertex of degree 2

in a branch that has both of its ends in  $C_1$ . But then by Claim applied to  $G_2$ ,  $G$  satisfies the theorem.

**Case 3:**  $|R| = 3$ .

Let  $R = \{y_1, y_2, y_3\}$ . First, let us suppose that both  $G_1$  and  $G_2$  have exactly two branch vertices, and that  $v_i$  is a branch vertex of  $G_i$  different from  $x$ , for  $i = 1, 2$ . If  $G_i$ , for  $i = 1, 2$ , does not have a vertex of degree 2 other than  $y_j$ , for  $j = 1, 2, 3$ , then  $G$  is a  $K_{3,3}$ , and hence it satisfies (iii) of the theorem. So, we may assume that there is a vertex of degree 2 (in  $G$ ) in a branch of  $G_1$  containing  $y_1$ . If  $y_2v_2$  or  $y_3v_2$  is not an edge, then  $G$  satisfies (i) of the theorem, so we may assume that  $y_2v_2$  and  $y_3v_2$  are edges. If  $y_1v_2$  is also an edge, then  $x$  and  $v_2$  form a pair of twins, and therefore  $G$  satisfies (ii) of the theorem. When  $y_1v_2$  is not an edge, then by symmetry  $v_1y_2$  and  $v_1y_3$  are edges. But then  $y_2$  and  $y_3$  form a pair of twins, and therefore  $G$  satisfies (ii) of the theorem.

Observe that if  $G_i$  has at least three branch vertices, then, by the choice of  $G$ , there is a vertex  $u_i$  of degree 2 in a branch of  $G_i$  not having  $x$  as its end, or  $G_i$  has a pair of twins that does not contain  $x$  (since  $G_i$  has at least three branch vertices). So if both  $G_1$  and  $G_2$  have at least three branch vertices, then the theorem holds. Therefore we may assume that  $G_1$  has at least three and  $G_2$  exactly two branch vertices. If  $G_2$  has a vertex  $u_2$  of degree 2 not in  $\{y_1, y_2, y_3\}$ , then  $G$  satisfies (i) or (ii) of the theorem. So we may assume that the only vertices of  $G_2$  of degree 2 are  $y_1, y_2$  and  $y_3$ , and therefore  $x$  and the other branch vertex of  $G_2$  form a pair of twins, hence  $G$  satisfies (ii) or (iii). This completes the proof when  $G$  has a star cutset.

**Proof when  $G$  has a 6-join.**

We may assume that  $G$  has no star cutset. In particular,  $G$  does not contain a pair of twins. Let  $(X_1, X_2, A_1, A_2, A_3, A_4, A_5, A_6)$  be a split of a 6-join of  $G$  and let  $A = \cup_{i=1}^6 A_i$ . By Lemma 3.2 (iii),  $|A_i| = 1$  for every  $i \in \{1, \dots, 6\}$  and all nodes of  $A$  are of degree 3 in  $G$ . It follows that both blocks of decomposition  $G_1$  and  $G_2$  have at least three branch vertices. By the choice of  $G$ , each of them has a vertex of degree 2 not in  $A$ , and hence  $G$  satisfies (i) of the theorem. This completes the proof.  $\square$

As a consequence of Theorem 5.1 we have the following corollary, a special case of which was conjectured in [13].

**Corollary 5.3** *If  $G$  is a cubic balanceable graph that is not  $R_{10}$ , then  $G$  has a pair of twins none of whose neighbours is a cut vertex of  $G$ .*

PROOF — Let  $G'$  be an end block of  $G$ . Then  $G'$  has at most one vertex of

degree 2, and all the other vertices of degree 3. If  $G'$  does not have a vertex of degree 2, then let  $G'' = G'$ , and otherwise let  $G''$  be the graph obtained from  $G'$  by subdividing twice an edge incident to the degree 2 vertex. Clearly  $G''$  is 2-connected balanceable and not equal to  $R_{10}$ . Note that  $G''$  has at most one branch of length greater than 1. By Theorem 5.1  $G''$  has a pair of twins  $\{u_1, u_2\}$ . Note that none of the neighbours of  $u_1$  and  $u_2$  in  $G''$  can be of degree 2 in  $G''$ , and hence  $\{u_1, u_2\}$  is the desired pair of twins of  $G$ .  $\square$

As it was noticed in [13] (for the special case of cubic balanced graphs), Corollary 5.3 implies the following.

**Corollary 5.4** *Let  $G$  be a cubic balanceable graph. Then the following hold:*

- (i)  $G$  has girth four.
- (ii) If  $G \neq R_{10}$  then  $G$  contains an edge that is not the unique chord of a cycle.
- (iii)  $G$  is not planar.

PROOF — It is easy to see that if  $G = R_{10}$  then (i) and (iii) hold. So we may assume that  $G \neq R_{10}$ . By Corollary 5.3, let  $\{u_1, u_2\}$  be a pair of twins of  $G$ , and  $\{v_1, v_2, v_3\}$  the set of neighbors of  $u_1$  and  $u_2$ . Then  $u_1v_1u_2v_2$  is a cycle of length 4, and hence (i) holds. Suppose that  $u_1v_1$  is a unique chord of a cycle  $C$  in  $G$ . Then all neighbors of  $u_1$  and  $v_1$  belong to  $C$ , and in particular,  $u_2$  belongs to  $C$  and has three neighbours in  $C$ , a contradiction. Hence (ii) holds.

By Corollary 5.3 we may assume that none of  $v_1, v_2, v_3$  is a cut vertex of  $G$ . So there is a connected component  $C$  of  $G \setminus \{u_1, u_2, v_1, v_2, v_3\}$  such that all of  $v_1, v_2, v_3$  have a neighbour in  $C$ . Let  $C'$  be a minimal induced subgraph of  $C$  that is connected and all of  $v_1, v_2, v_3$  have a neighbour in  $C'$ . Since  $G$  is cubic, it is easy to see that  $V(C') \cup \{u_1, u_2, v_1, v_2, v_3\}$  induces a subdivision of  $K_{3,3}$ . Therefore, by Kuratowski's Theorem (see for example [2]),  $G$  is not planar.  $\square$

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