Linear balanceable and subcubic balanceable graphs

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Co-authors

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Motivation

Conjecture (Morris, Spiga and Webb)

If G is cubic and every induced cycle has length divisible by 4, then G has a pair of twins.

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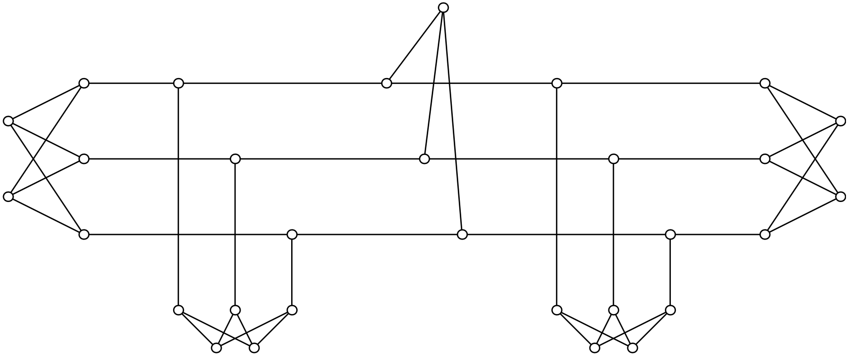


Definitions

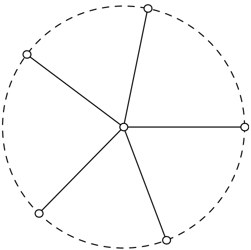
Definition

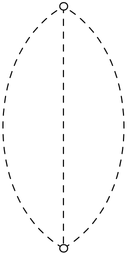
Let G be a bipartite graph, we say that G is *balanceable* if we can give weights $+1$, -1 to edges such that the weight of every induced cycle is divisible by 4.

Characterization

Theorem (Truemper)

A bipartite graph is balanceable if and only if it does not contain an odd wheel nor an odd 3-path configuration.

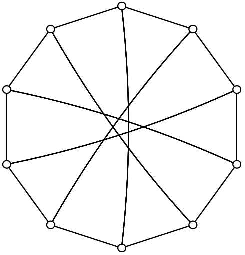


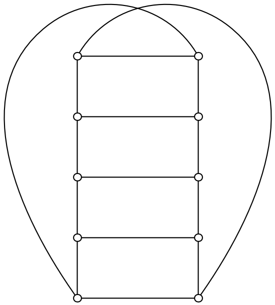


Conjecture

Conjecture (Conforti, Cornuéjols and Vušković)

In a balanceable bipartite graph either every edge belongs to some R_{10} or there is an edge that is not the unique chord of a cycle.





Main results

Theorem

If G is a 4-hole free balanceable graph on at least two vertices, then G contains at least two vertices of degree at most 2.

Theorem

If G is a cubic balanceable graph that is not R_{10}, then G has a pair of twins none of whose neighbors is a cut vertex of G.

Corollary

The conjecture is true if G does not contain a 4-hole or if $\Delta(G) \leq 3$.

Decomposition

Theorem (Conforti, Cornuéjols, Kappor and Vušković + Conforti and Rao + Yannakakis + easy lemma)

Let G be a connected balanceable graph.

- *If G is 4-hole free, then G is basic, or has a 2-join, a 6-join or a star cutset.*
- *If $\Delta(G) \leq 3$, then G is basic or is R_{10}, or has a 2-join, a 6-join or a star cutset.*

The Good

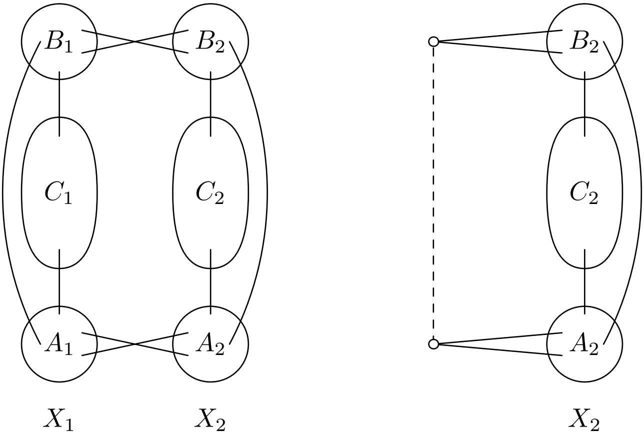


Figure : 2-join

The Bad

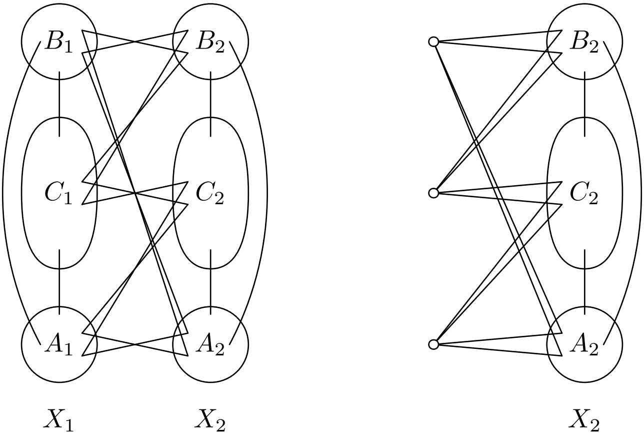


Figure : 6-join

The Ugly

Definition

A *star cutset* in a graph G is a set S of vertices such that:

- $G \setminus S$ is disconnected.
- S contains a vertex v adjacent to all other vertices of S.

We note (x, R) the star cutset.

In a perfect world

Theorem

Let G be bipartite 4-hole free with no-star cutset, then $\{2,6\}$-join blocks preserve:

- *Being balanceable;*
- *Having no star cutset;*
- *Having no 6-join.*

In a perfect world

Theorem

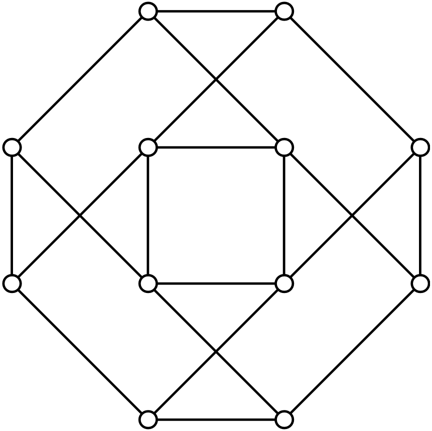
Let G be bipartite 4-hole free with no-star cutset, then $\{2,6\}$-join blocks preserve:

- *Being balanceable;*
- *Having no star cutset;*
- *Having no 6-join.*

Theorem

Let G be a bipartite 4-hole free graph. Let X_1, X_2 be a minimally-sided $\{2,6\}$-join. If G has no star cutset, then the block of decomposition G_1 has no $\{2,6\}$-join.

Crossing 2-join



Star cutset, again

Definition

A *star cutset* in a graph G is a set S of vertices such that:

- $G \setminus S$ is disconnected.
- S contains a vertex v adjacent to all other vertices of S.

Definition

A *double star cutset* in a graph G is a set S of vertices such that:

- $G \setminus S$ has two disconnected components C_1 and C_2.
- S contains an edge uv such that every vertex in S is adjacent to u or v.

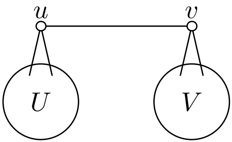
We call $C_1 \cup S$ and $C_2 \cup S$ the blocks of decomposition, and we note (u, v, U, V) where $U \subseteq N(u)$ and $V \subseteq N(v)$ the double star cutset.

Extreme double star cutset

Theorem

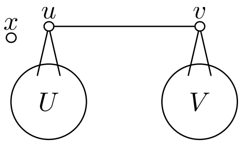
Let G be a 2-connected 4-hole free bipartite graph that has a star cutset. Let G_1 be a minimal side of a minimally-sided double star cutset of G. Then G_1 does not have a star cutset.

Extreme double star cutset



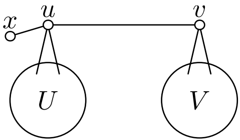
- G_1 is 2-connected.

Extreme double star cutset



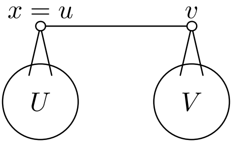
- G_1 is 2-connected.
- (x, R) a star cutset in G_1.
- $|R \cap S| \leq 1$.
- If $R \cap \{u, v\} = \emptyset$ then $(x, y \in R, R \setminus \{y\}, \emptyset)$ is a double star cutset in G.

Extreme double star cutset



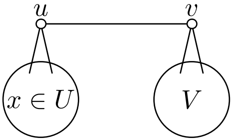
- G_1 is 2-connected.
- C component in $G_1 \setminus (\{x\} \cup R)$ with $C \cap (\{v\} \cup V) = \emptyset$.
- $C \setminus U \neq \emptyset$
- $(x, u, R \setminus \{u\}, U)$ is a double star cutset in G.

Extreme double star cutset



- If a component of $G_1 \setminus (\{x\} \cup R)$ contains a vertex from U or V, it contains vertex from $G_1 \setminus S$.
- $(x, v, U \cup R \setminus \{v\}, V)$ is a double star cutset in G.

Extreme double star cutset



- $\{v\} \cup V$ are in the same component in $G_1 \setminus (\{x\} \cup R)$
- If a component of $G_1 \setminus (\{x\} \cup R)$ contains a vertex from U, it contains vertex from $G_1 \setminus S$.
- $(x, u, R \setminus \{u\}, U \setminus \{x\})$ is a double star cutset in G.

Sketch of the proof

Theorem

If G is a 4-hole free balanceable graph on at least two vertices, then G contains at least two vertices of degree at most 2.

Proof.

- If we have a cut vertex it is easy.
- Assume there is a star cutset.
- Take a double star cutset such that the block G' has no star cutset.
- G' is basic or has $\{2, 6\}$-join.
- If G' is basic find two vertices of degree 2.
- Take (X_1, X_2) a minimally-sided $\{2, 6\}$-join with small intersection with the double star cutset.
- Now G_1 is basic, find good vertices in it.

□

Open questions

Question

How to build every cubic graph such that every induced cycle has length divisible by 4 ?

Conjecture (Conforti, Cornuéjols and Vušković)

In a balanceable bipartite graph either every edge belongs to some R_{10} or there is an edge that is not the unique chord of a cycle.

Thanks for you attention.