

# Clique Stable separation in perfect graphs with no balanced skew-partitions

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## Abstract

We prove that perfect graphs with no balanced skew-partitions admits a  $\mathcal{O}(n^2)$  Clique-Stable separator. We then prove that for every graph  $G$  in this class, there always exists two subsets of vertices  $V_1, V_2$  such that  $|V_1|, |V_2| \geq c|V(G)|$  for some constant  $c$  and  $V_1$  is either complete or anticomplete to  $V_2$ . The method uses trigraphs, introduced by Chudnovsky [5], and relies on the good behavior of both problems with respect to 2-join decomposition proved by Chudnovsky, Trotignon, Trunck and Vušković [7]. Moreover we define the generalized  $k$ -join and generalize both our results on classes of graphs admitting such a decomposition.

## 1 Introduction

In 1990, Yannakakis [12] studies the Vertex Packing polytope of a graph (also called Stable Set polytope), and asked for the existence of an extended formulation, that is to say a simpler polytope in higher dimension whose projection would be the vertex packing polytope. He then focuses on the case of perfect graphs, on which the non-negativity and the clique constraints suffice to describe the vertex packing polytope. This leads him to a communication complexity problem which can be restated as follows, where a *cut* is a bipartition of the vertices of the graph: does there exists a polynomial family  $F$  of cuts such that, for every clique  $K$  and every stable set  $S$  of the graph that do not intersect, there exists a cut  $(U, W)$  of  $F$  that *separates*  $K$  and  $S$ , meaning  $K \subseteq U$  and  $S \subseteq W$ ? Such a family of cuts separating all the cliques and the stable sets is called a Clique-Stable set separator (CS-separator for short). The existence of a polynomial CS-separator (called the *Clique-Stable separation*) is a necessary condition for the existence of an extended formulation. Yannakakis showed that both exist for several subclasses of perfect graphs, such as comparability graphs and their complements, chordal graphs and their complements, and Lovász proved it [11] for a generalization of series-parallel graphs called  $t$ -perfect graphs. However, they were unable to prove it for the whole class of perfect graphs.

Twenty years have past since Yannakakis introduced the problem and several results may enlighten the problem. First of all, a negative result due to Fiorini et al [8] asserts that there does not exists an extended formulation for the vertex packing polytope for all graphs. This pushes us further to the study of perfect graphs, for which great progress have been made. The most famous one is the Strong Perfect Graph Theorem, proving that a graph is perfect if and only if it is Berge. It was proved by Chudnovsky, Robertson, Seymour and Thomas [6], and their proof relies on a decomposition theorem: every Berge graph is either in some basic class, or has some kind of decomposition. Such results are hopefully helpful to answer Yannakakis questions on the

extended formulation and the Clique-Stable separation. However one decomposition, called the *balanced skew-partition* is notoriously difficult to handle. Nevertheless, can we still use the other decompositions ? Extending the notion of graphs to *trigraphs*, as it was done in the proof of the strong perfect graph theorem [5], Chudnovsky, Trotignon, Trunck and Vušković [7] proved that forbidding the balanced skew-partition inductively preserves some structure. Then they were able to use that structure to build a polynomial algorithm that colors Berge graphs with no balanced skew-partition.

Using the same decomposition results which we recall in Section 3, we prove in Section 4 that any Berge graph on  $n$  vertices with no balanced skew-partition admits a  $\mathcal{O}(n^2)$  CS-separator. We then generalize the method in Section 5 by defining the operation of *generalized  $k$ -join* between two trigraphs. We show that if  $\mathcal{C}$  is a class of graphs admitting a  $\mathcal{O}(n^c)$  CS-separator, its closure by *generalized  $k$ -join* (after an intermediate step changing  $\mathcal{C}$  into a class of trigraphs) admits a  $\mathcal{O}(n^{k^2c})$  CS-separator. This extends the list of graphs admitting a polynomial CS-separator, which has recently [2] been added of random graphs,  $H$ -free graphs if  $H$  is a split graph, and graphs with no induced path of length  $k$  and its complement. Finally, we prove in Section 6 that if  $G$  is a Berge graph with no balanced skew partition, there always exists two subsets of vertices  $V_1, V_2$  such that  $|V_1|, |V_2| \geq n/148$  and  $V_1$  is complete or anticomplete to  $V_2$ . This question is motivated by this property being false on some perfect graphs for any constant factor [9] and by its link with the Erdős-Hajnal property [1, 10]. We end up by generalizing this result to hereditary classes of trigraphs closed by generalized  $k$ -join if the property holds in the basic class considered.

## 2 Definitions

### 2.1 Trigraphs

For a set  $X$ , we denote by  $\binom{X}{2}$  the set of all subsets of  $X$  of size 2. For brevity of notation an element  $\{u, v\}$  of  $\binom{X}{2}$  is also denoted by  $uv$  or  $vu$ . A *trigraph*  $T$  consists of a finite set  $V(T)$ , called the *vertex set* of  $T$ , and a map  $\theta : \binom{V(T)}{2} \rightarrow \{-1, 0, 1\}$ , called the *adjacency function*.

Two distinct vertices of  $T$  are said to be *strongly adjacent* if  $\theta(uv) = 1$ , *strongly antiadjacent* if  $\theta(uv) = -1$ , and *semiadjacent* if  $\theta(uv) = 0$ . We say that  $u$  and  $v$  are *adjacent* if they are either strongly adjacent, or semiadjacent; and *antiadjacent* if they are either strongly antiadjacent, or semiadjacent. An *edge* (*antiedge*) is a pair of adjacent (antiadjacent) vertices. If  $u$  and  $v$  are adjacent (antiadjacent), we also say that  $u$  is *adjacent* (*antiadjacent*) to  $v$ , or that  $u$  is a *neighbor* (*antineighbor*) of  $v$ . The *open neighborhood*  $N(u)$  of  $u$  is the set of neighbors of  $u$ , and the *closed neighborhood*  $N[u]$  of  $u$  is  $N(u) \cup \{u\}$ . If  $u$  and  $v$  are strongly adjacent (strongly antiadjacent), then  $u$  is a *strong neighbor* (*strong antineighbor*) of  $v$ . Let  $\eta(T)$  be the set of all strongly adjacent pairs of  $T$ ,  $\nu(T)$  the set of all strongly antiadjacent pairs of  $T$ , and  $\sigma(T)$  the set of all semiadjacent pairs of  $T$ . Thus, a trigraph  $T$  is a graph if  $\sigma(T)$  is empty. A pair  $\{u, v\} \subseteq V(T)$  of distinct vertices is a *switchable pair* if  $\theta(uv) = 0$ , a *strong edge* if  $\theta(uv) = 1$  and a *strong antiedge* if  $\theta(uv) = -1$ . An edge  $uv$  (antiedge, strong edge, strong antiedge, switchable pair) is *between* two sets  $A \subseteq V(T)$  and  $B \subseteq V(T)$  if  $u \in A$  and  $v \in B$  or if  $u \in B$  and  $v \in A$ .

Let  $T$  be a trigraph. The *complement*  $\bar{T}$  of  $T$  is a trigraph with the same vertex set as  $T$ , and adjacency function  $\bar{\theta} = -\theta$ . Let  $A \subset V(T)$  and  $b \in V(T) \setminus A$ . We say that  $b$  is *strongly complete* to  $A$  if  $b$  is strongly adjacent to every vertex of  $A$ ;  $b$  is *strongly anticomplete* to  $A$  if  $b$  is strongly antiadjacent to every vertex of  $A$ ;  $b$  is *complete* to  $A$  if  $b$  is adjacent to every vertex of  $A$ ; and  $b$  is

*anticomplete* to  $A$  if  $b$  is antiadjacent to every vertex of  $A$ . For two disjoint subsets  $A, B$  of  $V(T)$ ,  $B$  is *strongly complete* (*strongly anticomplete*, *complete*, *anticomplete*) to  $A$  if every vertex of  $B$  is strongly complete (strongly anticomplete, complete, anticomplete) to  $A$ .

A *clique* in  $T$  is a set of vertices all pairwise adjacent, and a *strong clique* is a set of vertices all pairwise strongly adjacent. A *stable set* is a set of vertices all pairwise antiadjacent, and a *strongly stable set* is a set of vertices all pairwise strongly antiadjacent. For  $X \subseteq V(T)$  the trigraph *induced by  $T$  on  $X$*  (denoted by  $T|X$ ) has vertex set  $X$ , and adjacency function that is the restriction of  $\theta$  to  $\binom{X}{2}$ . Isomorphism between trigraphs is defined in the natural way, and for two trigraphs  $T$  and  $H$  we say that  $H$  is an *induced subtrigraph* of  $T$  (or  $T$  *contains  $H$  as an induced subtrigraph*) if  $H$  is isomorphic to  $T|X$  for some  $X \subseteq V(T)$ . Since in this paper we are only concerned with the induced subtrigraph containment relation, we say that  $T$  *contains  $H$*  if  $T$  contains  $H$  as an induced subtrigraph. We denote by  $T \setminus X$  the trigraph  $T|(V(T) \setminus X)$ .

Let  $T$  be a trigraph. A *path*  $P$  of  $T$  is a sequence of distinct vertices  $p_1, \dots, p_k$  such that either  $k = 1$ , or for  $i, j \in \{1, \dots, k\}$ ,  $p_i$  is adjacent to  $p_j$  if  $|i - j| = 1$  and  $p_i$  is antiadjacent to  $p_j$  if  $|i - j| > 1$ . Under these circumstances,  $V(P) = \{p_1, \dots, p_k\}$  and we say that  $P$  is a path *from  $p_1$  to  $p_k$* , its *interior* is the set  $P^* = V(P) \setminus \{p_1, p_k\}$ , and the *length* of  $P$  is  $k - 1$ . Sometimes, we denote  $P$  by  $p_1 \cdots p_k$ . Observe that, since a graph is also a trigraph, it follows that a path in a graph, the way we have defined it, is what is sometimes in literature called a chordless path.

A *hole* in a trigraph  $T$  is an induced subtrigraph  $H$  of  $T$  with vertices  $h_1, \dots, h_k$  such that  $k \geq 4$ , and for  $i, j \in \{1, \dots, k\}$ ,  $h_i$  is adjacent to  $h_j$  if  $|i - j| = 1$  or  $|i - j| = k - 1$ ; and  $h_i$  is antiadjacent to  $h_j$  if  $1 < |i - j| < k - 1$ . The *length* of a hole is the number of vertices in it. Sometimes we denote  $H$  by  $h_1 \cdots h_k$ . An *antipath* (*antihole*) in  $T$  is an induced subtrigraph of  $T$  whose complement is a path (hole) in  $\overline{T}$ .

A *semirealization* of a trigraph  $T$  is any trigraph  $T'$  with vertex set  $V(T)$  that satisfies the following: for all  $uv \in \binom{V(T)}{2}$ , if  $uv \in \eta(T)$  then  $uv \in \eta(T')$ , and if  $uv \in \nu(T)$  then  $uv \in \nu(T')$ . Sometimes we will describe a semirealization of  $T$  as an *assignment of values* to switchable pairs of  $T$ , with three possible values: “strong edge”, “strong antiedge” and “switchable pair”. A *realization* of  $T$  is any graph that is semirealization of  $T$  (so, any semirealization where all switchable pairs are assigned the value “strong edge” or “strong antiedge”). For  $S \subseteq \sigma(T)$ , we denote by  $G_S^T$  the realization of  $T$  with edge set  $\eta(T) \cup S$ , so in  $G_S^T$  the switchable pairs in  $S$  are assigned the value “edge”, and those in  $\sigma(T) \setminus S$  the value “antiedge”. The realization  $G_{\sigma(T)}^T$  is called the *full realization* of  $T$ .

Let  $T$  be a trigraph. For  $X \subseteq V(T)$ , we say that  $X$  and  $T|X$  are *connected* (*anticonnected*) if the graph  $G_{\sigma(T|X)}^{T|X}$  ( $G_{\emptyset}^{T|X}$ ) is connected. A *connected component* (or simply *component*) of  $X$  is a maximal connected subset of  $X$ , and an *anticonnected component* (or simply *anticomponent*) of  $X$  is a maximal anticonnected subset of  $X$ .

A trigraph  $T$  is *Berge* if it contains no odd hole and no odd antihole. Therefore, a trigraph is Berge if and only if its complement is. We observe that  $T$  is Berge if and only if every realization (semirealization) of  $T$  is Berge.

Finally let us define the class of trigraph we are working on. Let  $T$  be a trigraph, denote by  $\Sigma(T)$  the graph with vertex set  $V(T)$  and edge set  $\sigma(T)$  (the switchable pairs of  $T$ ). The connected components of  $\Sigma(T)$  are called the *switchable components* of  $T$ . Let  $\mathcal{F}$  be the class of Berge trigraphs such that the following hold:

- Every switchable component of  $T$  has at most two edges (and therefore no vertex has more than two neighbors in  $\Sigma(T)$ ).

- Let  $v \in V(T)$  have degree two in  $\Sigma(T)$ , denote its neighbors by  $x$  and  $y$ . Then either  $v$  is strongly complete to  $V(T) \setminus \{v, x, y\}$  in  $T$ , and  $x$  is strongly adjacent to  $y$  in  $T$ , or  $v$  is strongly anticomplete to  $V(T) \setminus \{v, x, y\}$  in  $T$ , and  $x$  is strongly antiadjacent to  $y$  in  $T$ .

Observe that  $T \in \mathcal{F}$  if and only if  $\bar{T} \in \mathcal{F}$ .

## 2.2 Clique-Stable set separation

Let  $T$  be a trigraph. A *cut* is a pair  $(U, W) \subseteq V(T)^2$  such that  $U \cup W = V(T)$  and  $U \cap W = \emptyset$ . It *separates* a clique  $K$  and a stable set  $S$  if  $K \subseteq U$  and  $S \subseteq W$ . Sometimes we will call  $U$  the *clique side* of the cut and  $W$  the *stable set side* of the cut. Note that a clique and a stable set can be separated if and only if they do not intersect. Moreover note that they can intersect only on a switchable component  $V$  containing only switchable pairs, i.e. such that for every  $u, v \in V$ ,  $u = v$  or  $uv \in \sigma(T)$ . In particular if  $T \in \mathcal{F}$ , a clique and a stable set can intersect on at most a vertex or a switchable pair. We say that a family  $F$  of cuts is a *CS-separator* if for all every  $K$  and every stable set  $S$  which do not intersect, there exists a cut in  $F$  that separates  $K$  and  $S$ . Given a class  $\mathcal{C}$  of trigraphs, the interesting question is to know whether there exists a constant  $c$  such that every trigraph of  $\mathcal{C}$  admits a CS-separator of size  $\mathcal{O}(n^c)$ .

Suppose there exists a CS-separator of size  $m$  on  $T$ , then we build a CS-separator of size  $m$  on  $\bar{T}$  by building for every cut  $(U, W)$  the cut  $(W, U)$ . Thus, the problem is auto-complementary. Moreover, let us show that we can only focus on maximal cliques and stable sets.

**Lemma 2.1.** *If a trigraph  $T$  of  $\mathcal{F}$  admits a family  $F$  of cuts separating all the maximal (in the sense of inclusion) cliques from the maximal stable sets, then it admits a CS-separator of size at most  $|F| + \mathcal{O}(n^2)$ .*

*Proof.* For every  $x \in V$ , let  $Cut_{1,x}$  be the cut  $(N[x], V \setminus N[x])$  and  $Cut_{2,x}$  be the cut  $(N(x), V \setminus N(x))$ . For every switchable pair  $xy$ , let  $Cut_{1,xy}$  (resp.  $Cut_{2,xy}, Cut_{3,xy}, Cut_{4,xy}$ ) be the cut  $(U = N[x] \cup N[y], V \setminus U)$  (resp.  $(U = N[x] \cup N(y), V \setminus U)$ ,  $(U = N(x) \cup N[y], V \setminus U)$ ,  $(U = N(x) \cup N(y), V \setminus U)$ ). Let  $F'$  be the union of  $F$  with all these cuts for every  $x \in V$ ,  $xy \in \sigma(T)$ , and let us prove that  $F'$  is a CS-separator.

Let  $(K, S)$  be a pair of clique and stable set that do not intersect. Extend  $K$  and  $S$  by adding vertices to get a maximal clique  $K'$  and a maximal stable set  $S'$ . Three cases are to be considered. Either  $K'$  and  $S'$  do not intersect, and there is a cut in  $F$  that separates  $K'$  from  $S'$  (thus  $K$  from  $S$ ). Or  $K'$  and  $S'$  intersect on a vertex  $x$  : if  $x \in K$ , then  $Cut_{1,x}$  separates  $K$  from  $S$ , otherwise  $Cut_{2,x}$  does. Or else  $K'$  and  $S'$  intersect on a switchable pair  $xy$  (recall that a clique and a stable set can intersect on at most one vertex or one switchable pair): then the same argument can be applied with  $Cut_{1,xy}$ , or  $\dots$ , or  $Cut_{4,xy}$  depending on the intersection between  $\{x, y\}$  and  $K$ .  $\square$

In particular, if  $T$  has at most  $\mathcal{O}(n^c)$  maximal cliques (or stable sets) for some constant  $c \geq 2$ , then there is a CS-separator of size  $\mathcal{O}(n^c)$  (we just cut every maximal clique from the rest of the graph and then apply the previous Lemma).

## 3 Decomposing trigraphs of $\mathcal{F}$

This section recall definitions and results from [7] that we use in the next section. Our goal is to state the decomposition theorem for trigraphs of  $\mathcal{F}$  by building the blocks of decomposition and stating the theorems showing the blocks stay in the class. First we need some definitions.

### 3.1 Basic trigraphs

A trigraph  $T$  is *bipartite* if its vertex set can be partitioned into two strongly stable sets. A trigraph  $T$  is a *line trigraph* if the full realization of  $T$  is the line graph of a bipartite graph and every clique of size at least 3 in  $T$  is a strong clique. Let us now define the trigraph analogue of the double split graph (first defined in [6]), namely the *doubled trigraph*. A *good partition* of a trigraph  $T$  is a partition  $(X, Y)$  of  $V(T)$  (possibly,  $X = \emptyset$  or  $Y = \emptyset$ ) such that:

- Every component of  $T|X$  has at most two vertices, and every anticomponent of  $T|Y$  has at most two vertices.
- No switchable pair of  $T$  meets both  $X$  and  $Y$ .
- For every component  $C_X$  of  $T|X$ , every anticomponent  $C_Y$  of  $T|Y$ , and every vertex  $v$  in  $C_X \cup C_Y$ , there exists at most one strong edge and at most one strong antiedge between  $C_X$  and  $C_Y$  that is incident to  $v$ .

A trigraph is *doubled* if it has a good partition. A trigraph is *basic* if it is either a bipartite trigraph, the complement of a bipartite trigraph, a line trigraph, the complement of a line trigraph or a doubled trigraph. Basic trigraphs behave well with respect to induced subtrigraphs and complementation as stated by the following lemma.

**Lemma 3.1** ([7]). *Basic trigraphs are Berge and are closed under taking induced subtrigraphs, semirealizations, realizations and complementation.*

### 3.2 Decompositions

We now describe the decompositions that we need to state the decomposition theorem. First, a *2-join* in a trigraph  $T$  is a partition  $(X_1, X_2)$  of  $V(T)$  such that there exist disjoint sets  $A_1, B_1, C_1, A_2, B_2, C_2 \subseteq V(T)$  satisfying:

- $X_1 = A_1 \cup B_1 \cup C_1$  and  $X_2 = A_2 \cup B_2 \cup C_2$ ;
- $A_1, A_2, B_1$  and  $B_2$  are non-empty;
- no switchable pair meets both  $X_1$  and  $X_2$ ;
- every vertex of  $A_1$  is strongly adjacent to every vertex of  $A_2$ , and every vertex of  $B_1$  is strongly adjacent to every vertex of  $B_2$ ;
- there are no other strong edges between  $X_1$  and  $X_2$ ;
- for  $i = 1, 2$   $|X_i| \geq 3$ ; and
- for  $i = 1, 2$ , if  $|A_i| = |B_i| = 1$ , then the full realization of  $T|X_i$  is not a path of length two joining the members of  $A_i$  and  $B_i$ .

In these circumstances, we say that  $(A_1, B_1, C_1, A_2, B_2, C_2)$  is a *split* of  $(X_1, X_2)$ . The 2-join is *proper* if for  $i = 1, 2$ , every component of  $T|X_i$  meets both  $A_i$  and  $B_i$ . Note that the fact that a 2-join is proper does not depend on the particular split that is chosen. A *complement 2-join* of a trigraph  $T$  is a 2-join of  $\overline{T}$ . In our class we have the following useful technical lemma.

**Lemma 3.2** ([7]). *Let  $T$  be a trigraph from  $\mathcal{F}$  with no balanced skew-partition, and let  $(A_1, B_1, C_1, A_2, B_2, C_2)$  be a split of a 2-join  $(X_1, X_2)$  in  $T$ . Then  $|X_i| \geq 4$ , for  $i = 1, 2$ .*

In Berge trigraphs, 2-joins are *odd* or *even* according to the parity of the lengths of the paths between  $A_i$  and  $B_i$ . The following lemma ensure the correctness of the definition.

**Lemma 3.3.** *Let  $T$  be a Berge trigraph and  $(A_1, B_1, C_1, A_2, B_2, C_2)$  a split of a proper 2-join of  $T$ . Then all paths with one end in  $A_i$ , one end in  $B_i$  and interior in  $C_i$ , for  $i = 1, 2$ , have lengths of the same parity.*

*Proof.* Otherwise, for  $i = 1, 2$ , let  $P_i$  be a path with one end in  $A_i$ , one end in  $B_i$  and interior in  $C_i$ , such that  $P_1$  and  $P_2$  have lengths of different parity. They form an odd hole, a contradiction.  $\square$

Our second decomposition is the balanced skew-partition. Let  $A, B$  be disjoint subsets of  $V(T)$ . We say the pair  $(A, B)$  is *balanced* if there is no odd path of length greater than 1 with ends in  $B$  and interior in  $A$ , and there is no odd antipath of length greater than 1 with ends in  $A$  and interior in  $B$ . A *skew-partition* is a partition  $(A, B)$  of  $V(T)$  so that  $A$  is not connected and  $B$  is not anticonnected. A skew-partition  $(A, B)$  is *balanced* if the pair  $(A, B)$  is.

These decompositions we just described generalize the decompositions used in [6], and in addition all the “important” edges and non-edges in those graph decompositions are required to be strong edges and strong antiedges of the trigraph, respectively. We are now ready to state the decomposition theorem.

**Theorem 3.4** ([7]). *Every trigraph in  $\mathcal{F}$  is either basic, or admits a balanced skew-partition, a proper 2-join, or a proper 2-join in the complement.*

We now define the *blocks of decomposition*  $T_X$  with respect to a 2-join (an illustration of blocks of decomposition can be found in Figure 2 in Section 4 where they are used with additional features, namely weights and labels).

If  $(X_1, X_2)$  is a proper odd 2-join and  $X = X_1$ , then let  $(A_1, B_1, C_1, A_2, B_2, C_2)$  be a split of  $(X_1, X_2)$ . We build the block of decomposition  $T_{X_1} = T_X$  as follows. We start with  $T|(A_1 \cup B_1 \cup C_1)$ . We then add two new *marker vertices*  $a$  and  $b$  such that  $a$  is strongly complete to  $A_1$ ,  $b$  is strongly complete to  $B_1$ ,  $ab$  is a switchable pair, and there are no other edges between  $\{a, b\}$  and  $X_1$ . Note that  $\{a, b\}$  is a switchable component of  $T_X$ .

If  $(X_1, X_2)$  is a proper even 2-join and  $X = X_1$ , then let  $(A_1, B_1, C_1, A_2, B_2, C_2)$  be a split of  $(X_1, X_2)$ . We build the block of decomposition  $T_{X_1} = T_X$  as follows. We start with  $T|(A_1 \cup B_1 \cup C_1)$ . We then add three new *marker vertices*  $a, b$  and  $c$  such that  $a$  is strongly complete to  $A_1$ ,  $b$  is strongly complete to  $B_1$ ,  $ac$  and  $cb$  are switchable pairs, and there are no other edges between  $\{a, b, c\}$  and  $X_1$ .

We define the block of decomposition of a complement 2-join  $(X_1, X_2)$  as the complement of the block of decomposition of the 2-join  $(X_1, X_2)$  in the complement of the original trigraph.

The two following theorems ensure that our blocks of decomposition stay in the class, namely are in  $\mathcal{F}$  and do not have balanced skew-partition.

**Theorem 3.5** ([7]). *If  $X$  is a proper 2-join or a complement of a proper 2-join of a trigraph  $T$  from  $\mathcal{F}$  with no balanced skew-partition, then  $T_X$  is a trigraph from  $\mathcal{F}$  with no balanced skew-partition.*

## 4 Clique-Stable separation in Berge graphs with no balanced skew partition

This part is devoted to proving that the trigraphs of  $\mathcal{F}$  with no balanced skew-partition admits a quadratic CS-separator. First, we handle the case of basic trigraphs:

**Lemma 4.1.** *There exists a constant  $c$  such that every basic trigraph admits a CS-separator of size  $cn^2$ .*

*Proof.* Since the problem is auto-complementary, we consider only the cases of bipartite graphs, line trigraphs and doubled trigraphs. A clique in a bipartite trigraph is an edge, a switchable pair or a vertex, thus there is at most a quadratic number of them. If  $T$  is a line trigraph, then its full realization is the line graph of a bipartite graph  $G$  thus  $T$  has a linear number of maximal cliques because each of which corresponds to a vertex of  $G$ . Thanks to Lemma 2.1, this implies the existence of a CS-separator of quadratic size.

If  $T$  is a doubled trigraph, let  $(X, Y)$  be a good partition of it and build the cut  $(Y, X)$  together with the following ones: for every  $Z = \{x\}$  with  $x \in X$  or  $Z = \emptyset$ , and for every  $Z' = \{y\}$  with  $y \in Y$  or  $Z' = \emptyset$ , take the cut  $(Y \cup Z \setminus Z', X \cup Z' \setminus Z)$  and for every pair  $x, y \in V$ , take the cut  $(\{x, y\}, V \setminus \{x, y\})$ , and  $(V \setminus \{x, y\}, \{x, y\})$ . These cuts form a CS-separator : let  $(K, S)$  be a clique and a stable set of  $T$  that do not intersect, then  $|K \cap X| \leq 2$  and  $|S \cap Y| \leq 2$ . If  $|K \cap X| = 2$  (resp.  $|S \cap Y| = 2$ ) then  $K$  (resp.  $S$ ) is just an edge (resp. an antiedge) because by definition, the vertices of  $K \cap X$  can not have common neighbors in  $Y$ . So the cut  $(K, V \setminus K)$  (resp.  $(V \setminus S, S)$ ) separates  $K$  and  $S$ . Otherwise,  $|K \cap X| \leq 1$  and  $|S \cap Y| \leq 1$  and then  $(Y \cup (K \cap X) \setminus (S \cap Y), X \cup (S \cap Y) \setminus (K \cap X))$  separates them.  $\square$

Next, we handle the case of a 2-join in the trigraph and show how to reconstruct a CS-separator from the CS-separators on the blocks of decompositions.

**Lemma 4.2.** *Let  $T$  be a trigraph admitting a proper 2-join  $(X_1, X_2)$ . If the blocks of decomposition  $T_{X_1}$  and  $T_{X_2}$  admits a CS-separator of size respectively  $k_1$  and  $k_2$ , then  $T$  admits a CS-separator of size  $k_1 + k_2$ .*

*Proof.* Let  $(A_1, B_1, C_1, A_2, B_2, C_2)$  be a split of  $(X_1, X_2)$ ,  $T_{X_1}$  (resp.  $T_{X_2}$ ) the block of decomposition with marker vertices  $a_1, b_1$ , and possibly  $c_1$  (depending on the parity of the 2-join) (resp.  $a_2, b_2$ , and possibly  $c_2$ ). Observe that there is no need to distinguish between an odd or an even 2-join, because  $c_1$  and  $c_2$  play no role. Let  $F_1$  be a CS-separator of  $T_{X_1}$  of size  $k_1$  and  $F_2$  be a CS-separator of  $T_{X_2}$  of size  $k_2$ . Let us build  $F$  aiming at being a CS-separator for  $T$ . For each cut  $(U, W) \in F_1$ , build the cut  $((U \cap X_1) \cup U', (W \cap X_1) \cup W' \cup C_2)$  where  $U' \cup W' = A_2 \cup B_2$  and  $A_2 \subseteq U'$  (resp.  $B_2 \subseteq U'$ ) if  $a_2 \in U$  (resp.  $b_2 \in U$ ), and  $A_2 \subseteq W'$  (resp.  $B_2 \subseteq W'$ ) otherwise. In other words, we put  $A_2$  on the same side as  $a_2$ ,  $B_2$  on the same side as  $b_2$ , and  $C_2$  on the stable set side. For each cut in  $F_2$ , we do the similar construction: put  $A_1$  on the same side as  $a_1$ ,  $B_1$  on the same side as  $b_1$ , and finally put  $C_1$  on the stable set side.

$F$  is indeed a CS-separator: let  $(K, S)$  be a pair of clique and stable set that do not intersect. As a first case, suppose that  $K \subseteq X_1$ . Let  $S' = (S \cap X_1) \cup S_{a_2, b_2}$  where  $S_{a_2, b_2} \subseteq \{a_2, b_2\}$  contains  $a_2$  (resp.  $b_2$ ) if and only if  $S$  intersects  $A_2$  (resp.  $B_2$ ).  $S'$  is a stable set of  $T_{X_1}$ , so there is a cut in  $F_1$  separating the pair  $(K, S')$ . The corresponding cut in  $F$  separates  $(K, S)$ . The case  $K \subseteq X_2$  is handled symmetrically.

As a second case, suppose  $K$  intersects both  $X_1$  and  $X_2$ . Then  $K \cap C_1 = \emptyset$  and  $K \subseteq A_1 \cup A_2$  or  $K \subseteq B_1 \cup B_2$ . Assume by symmetry that  $K \subseteq A_1 \cup A_2$ . Observe that  $S$  can not intersect both  $A_1$  and  $A_2$  which are completely strongly adjacent, say it does not intersect  $A_2$  (the other case is handled symmetrically). Let  $K' = (K \cap A_1) \cup \{a_2\}$  and  $S' = (S \cap X_1) \cup S_{b_2}$  where  $S_{b_2} = \{b_2\}$  if  $S$  intersects  $B_2$ , and  $S_{b_2} = \emptyset$  otherwise.  $K'$  is a clique and  $S'$  is a stable set of  $T_{X_1}$  so there exists a cut in  $F_1$  separating them, and the corresponding cut in  $F$  separates  $(K, S)$ . Then  $F$  is a CS-separator.  $\square$

This lead us to the main theorem of this part:

**Theorem 4.3.** *Every trigraph of  $\mathcal{F}$  with no balanced skew-partition admits a CS-separator of size  $\mathcal{O}(n^2)$ .*

*Proof.* Let  $c'$  be the constant of Lemma 4.1 and  $c = \max(c', 2^{24})$ . Let us prove by induction that every trigraph of  $T$  admits a CS-separator of size  $cn^2$ . The initialization is concerned with basic trigraphs, for which Lemma 4.1 shows that a CS-separator of size  $c'n^2$  exists, and with trigraphs of size less than 24. For them, one can consider every subset  $U$  of vertices and take the cut  $(U, V \setminus U)$  which form a trivial CS-separator of size at most  $2^{24}n^2$ .

Consequently, we can now assume that the trigraph  $T$  is not basic and has at least 25 vertices. Applying Theorem 3.4,  $T$  has a proper 2-join  $(X_1, X_2)$  (or the complement of a proper 2-join, in which case we solve on  $\bar{T}$  since the problem is auto-complementary). Let  $n_1 = |X_1|$ , by Lemma 3.2 we can assume that  $4 \leq n_1 \leq n - 4$ . Applying Theorem 3.5, we can apply the induction hypothesis on the blocks of decomposition  $T_{X_1}$  and  $T_{X_2}$  to get a CS-separator of size respectively at most  $k_1 = c(n_1 + 3)^2$  and  $k_2 = c(n - n_1 + 3)^2$ . By Lemma 4.2,  $T$  admits a CS-separator of size  $k_1 + k_2$ . The goal is to prove that  $k_1 + k_2 \leq cn^2$ .

Let  $P(n_1) = c(n_1 + 3)^2 + c(n - n_1 + 3)^2 - cn^2$ .  $P$  is a degree 2 polynomial with leading coefficient  $2c > 0$ . Moreover,  $P(4) = P(n - 4) = -2c(n - 25) \leq 0$  so by convexity of  $P$ ,  $P(n_1) \leq 0$  for every  $4 \leq n_1 \leq n - 4$ , which achieves the proof.  $\square$

## 5 Closure by generalized $k$ -join

We present here a way to extend the result of the Clique-Stable separation on Berge graphs with no balanced skew partition to larger classes of graphs, based on a generalization of the 2-join. Let  $\mathcal{C}$  be a class of graphs, which should be seen as "basic" graphs. For any integer  $k \geq 1$ , we construct the class  $\mathcal{C}^{\leq k}$  of trigraphs in the following way : a trigraph  $T$  belongs to  $\mathcal{C}^{\leq k}$  if and only if there exists a partition  $X_1, \dots, X_r$  of  $V(T)$  such that

- for every  $1 \leq i \leq r$ ,  $1 \leq |X_i| \leq k$ .
- for every  $1 \leq i \leq r$ ,  $\binom{X_i}{2} \subseteq \sigma(T)$ .
- for every  $1 \leq i \neq j \leq r$ ,  $X_i \times X_j \cap \sigma(T) = \emptyset$ .
- there exists a graph  $G$  in  $\mathcal{C}$  such that  $G$  is a realization of  $T$ .

In other words, starting from a graph  $G$  of  $\mathcal{C}$ , we partition its vertices into small parts (of size at most  $k$ ), and change all adjacency inside the parts into switchable pairs.



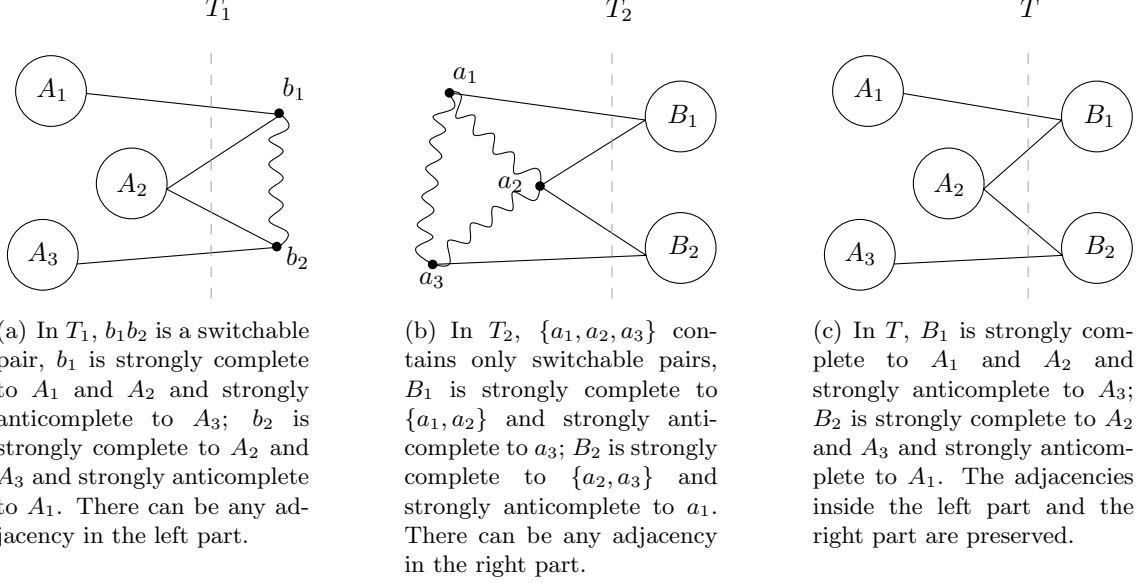


Figure 1: Example of a generalized 3-join  $T$  of  $T_1$  and  $T_2$  with  $r = 3$  and  $s = 2$ .

We now define the *generalized  $k$ -join* between two trigraphs  $T_1$  and  $T_2$  (see Figure 1 for an illustration), which generalize the 2-join and is similar to the  $H$ -join [4]. Let  $T_1$  and  $T_2$  be two trigraphs having the following properties with  $1 \leq r, s \leq k$ :

- $V(T_1)$  is partitioned into  $(A_1, \dots, A_r, B = \{b_1, \dots, b_s\})$  and  $A_j \neq \emptyset$  for every  $1 \leq j \leq r$ .
- $V(T_2)$  is partitioned into  $(B_1, \dots, B_s, A = \{a_1, \dots, a_r\})$  and  $B_i \neq \emptyset$  for every  $1 \leq i \leq s$ .
- $\binom{B}{2} \subseteq \sigma(T_1)$  and  $\binom{A}{2} \subseteq \sigma(T_2)$ , meaning that  $A$  and  $B$  contain only switchable pairs.
- For every  $1 \leq i \leq s, 1 \leq j \leq r$ ,  $b_i$  and  $a_j$  are either both strongly complete or both strongly anticomplete to respectively  $A_j$  and  $B_i$ . In other words, there exists a bipartite graph describing the adjacency between  $B$  and  $(A_1, \dots, A_r)$ , and the same bipartite graph describes the adjacency between  $(B_1, \dots, B_s)$  and  $A$ .

Then the generalized  $k$ -join of  $T_1$  and  $T_2$  is the trigraph  $T$  where  $V(T) = A_1 \cup \dots \cup A_r \cup B_1 \cup \dots \cup B_s$ . Let  $\theta_1$  and  $\theta_2$  be the adjacency function of  $T_1$  and  $T_2$ , respectively. As much as possible, the adjacency function  $\theta$  of  $T$  follows  $\theta_1$  and  $\theta_2$  (meaning  $\theta(uv) = \theta_1(uv)$  for  $uv \in \binom{V(T_1) \cap V(T)}{2}$  and  $\theta(uv) = \theta_2(uv)$  for  $uv \in \binom{V(T_2) \cap V(T)}{2}$ ), and for  $a \in A_j, b \in B_i$ ,  $\theta(ab) = 1$  if  $b_i$  and  $A_j$  are strongly complete in  $T_1$  (or, equivalently, if  $a_j$  and  $B_i$  are strongly complete in  $T_2$ ), and  $-1$  otherwise.

We finally define  $\overline{\mathcal{C}^{\leq k}}$  to be the smallest class containing  $\mathcal{C}^{\leq k}$  and closed under generalized  $k$ -join.

**Lemma 5.1.** *If every graph  $G$  of  $\mathcal{C}$  admits a CS-separator of size  $m$ , then every trigraph  $T$  of  $\mathcal{C}^{\leq k}$  admits a CS-separator of size  $m^{k^2}$ .*

*Proof.* First we claim that if there exist a CS-separator  $F$  of size  $m$  then  $F' = \{(\cap_{i=1}^k U_i, \cup_{i=1}^k W_i) \mid (U_1, W_1) \dots (U_k, W_k) \in F\}$  is a family of cuts of size  $m^k$  that separates every

clique from every union of at most  $k$  stable sets. Indeed if  $K$  is a clique and  $S_1 \dots S_k$  are  $k$  stable sets such that they do not intersect  $K$  then there exists in  $F$   $k$  partitions  $(U_1, W_1) \dots (U_k, W_k)$  such that  $(U_i, W_i)$  separates  $K$  and  $S_i$ . Now  $(\cap_{i=1}^k U_i, \cup_{i=1}^k W_i)$  is a partition that separates  $K$  from  $\cup_{i=1}^k S_i$ . Using the same argument we can build  $F''$  a family of cuts of size  $m^{k^2}$  that separates every union of at most  $k$  cliques from every union of at most  $k$  stable sets. Now let  $T$  be a trigraph of  $\mathcal{C}^{\leq k}$  and let  $G \in \mathcal{C}$  such that  $T$  is a realization of  $G$ . Notice that a clique  $K$  (resp. stable set  $S$ ) in  $T$  is an union of at most  $k$  cliques (resp. stable sets) in  $G$ . For instance one can see that  $\Sigma(T) \cap K$  (resp.  $\Sigma(T) \cap S$ ) is  $k$ -colorable and each class of color corresponds to a clique (resp. stable set) in  $G$ . Then there exists a CS-separator of  $T$  of size  $m^{k^2}$ .  $\square$

**Lemma 5.2.** *If  $T_1$  and  $T_2 \in \overline{\mathcal{C}^{\leq k}}$  admit CS-separators of size respectively  $m_1$  and  $m_2$ , then the generalized  $k$ -join  $T$  of  $T_1$  and  $T_2$  admits a CS-separator of size  $m_1 + m_2$ .*

*Proof.* The proof is very similar to the one of Lemma 4.2. We follow the notations introduced in the definition of the generalized  $k$ -join. Let  $F_1$  (resp.  $F_2$ ) be a CS-separator of size  $m_1$  (resp.  $m_2$ ) on  $T_1$  (resp.  $T_2$ ). Let us build  $F$  aiming at being a CS-separator on  $T$ . For every cut  $(U, W)$  in  $F_1$ , build the cut  $(U', W')$  according to the following two rules: for every  $a \in \cup_{j=0}^r A_j$  (resp.  $b \in B_i$ ),  $a \in U'$  (resp.  $b \in U'$ ) if and only if  $a \in U$  (resp.  $b_i \in U$ ). In other words, we take a cut similar to  $(U, W)$  by putting  $B_i$  in the same side as  $b_i$ . We do the symmetric operation for every cut  $(U, W)$  in  $F_2$  by putting  $A_j$  in the same side as  $a_j$ .

$F$  is indeed a CS-separator: let  $(K, S)$  be a pair of clique and stable set that do not intersect. Suppose as a first case that one part of the partition  $(A_1, \dots, A_r, B_1, \dots, B_s)$  intersects both  $K$  and  $S$ . Free to exchange  $T_1$  and  $T_2$ , and to renumber the  $A_j$ , we can assume that  $A_1 \cap K \neq \emptyset$  and  $A_1 \cap S \neq \emptyset$ . Since for every  $i$ ,  $A_1$  is either strongly complete or strongly anticomplete to  $B_i$ ,  $B_i$  can not intersect both  $K$  and  $S$ . Consider in  $T_1$  the cut  $(K' = (K \cap V(T)) \cup K_b, S' = (S \cap V(T)) \cup S_b)$  where  $K_b = \{b_i | K \cap B_i \neq \emptyset\}$  and  $S_b = \{b_i | S \cap B_i \neq \emptyset\}$ .  $K'$  is a clique in  $T_1$ ,  $S'$  is a stable set in  $T_1$ , and there is a cut separating them in  $F_1$ . The corresponding cut in  $F$  separates  $K$  and  $S$ .

The other case occurs when no part of the partition intersects both  $K$  and  $S$ . Then for every  $i$ ,  $B_i$  does not: the same arguments as before apply.  $\square$

**Theorem 5.3.** *If every graph of  $\mathcal{C}$  admits a CS-separator of size  $\mathcal{O}(n^c)$ , then every trigraph of  $\overline{\mathcal{C}^{\leq k}}$  admits a CS-separator of size  $\mathcal{O}(n^{k^2c})$ . In particular, every realization of a trigraph of  $\overline{\mathcal{C}^{\leq k}}$  admits a CS-separator of size  $\mathcal{O}(n^{k^2c})$ .*

*Proof.* We prove by induction that there exists a CS-separator of size  $pn^{k^2c}$  with  $p = \max(p', 2^{p_0})$  where  $p'$  is the constant in the  $\mathcal{O}$  of the size of a CS-separator of graphs in  $\mathcal{C}$  and  $p_0$  is a constant to be defined later. The base case is divided into two cases: the trigraphs of  $\mathcal{C}^{\leq k}$ , for which the property is verified according to Lemma 5.1, and the trigraphs of size at most  $p_0$ . For those, one can consider every subset  $U$  of vertices and take the cut  $(U, V \setminus U)$  which form a trivial CS-separator of size at most  $2^{p_0}n^{k^2c}$ .

Consequently, we can now assume that  $T$  is the generalized  $k$ -join of  $T_1$  and  $T_2$  with at least  $p_0$  vertices. Let  $n_1 = |T_1|$  and  $n_2 = |T_2|$  with  $n_1 + n_2 = n + r + s$  and  $r + s + 1 \leq n_1, n_2, \leq n - 1$ . By induction, there exists a CS-separator of size  $pn_1^{k^2c}$  on  $T_1$  and one of size  $pn_2^{k^2c}$  on  $T_2$ . By Lemma 5.2, there exists a CS-separator on  $T$  of size  $pn_1^{k^2c} + pn_2^{k^2c}$ . The goal is to prove  $pn_1^{k^2c} + pn_2^{k^2c} \leq pn^{k^2c}$ .

Notice that  $n_1 + n_2 = n - 1 + r + s + 1$  so by convexity of  $x \mapsto x^c$  on  $\mathbb{R}^+$ ,  $n_1^{k^2c} + n_2^{k^2c} \leq (n - 1)^{k^2c} + (r + s + 1)^{k^2c}$ . Moreover,  $r + s + 1 \leq 2k + 1$ . Now we can define  $p_0$  large enough such

that for every  $n \geq p_0$ ,  $n^{k^2c} - (n-1)^{k^2c} \geq (2k+1)^{k^2c}$ . Then  $n_1^{k^2c} + n_2^{k^2c} \leq n^{k^2c}$ , which concludes the proof.  $\square$

## 6 Linear strongly complete or anticomplete pseudo-bipartite subgraphs

### 6.1 On Berge graphs with no balanced skew partition

Given a constant  $c < 1$ , a trigraph  $T$  of size  $n$  has a  $c.n$  strongly complete or anticomplete pseudo-bipartite subgraph if there exists  $V_1, V_2 \subseteq V(T)$  such that  $|V_1|, |V_2| \geq c.n$  and  $V_1$  is strongly complete or strongly anticomplete to  $V_2$ . The aim of this part is to prove the following theorem:

**Theorem 6.1.** *Let  $T$  be a trigraph of  $\mathcal{F}$  with no balanced skew partition, such that  $n = |V(T)| \geq 3$ . Then  $T$  has a  $n/148$  strongly complete or anticomplete pseudo-bipartite subgraph.*

Let  $\mathcal{C}$  be a class of graphs closed under taking induced subgraphs, such that every  $G \in \mathcal{C}$  has a  $c.n$  strongly complete or anticomplete pseudo-bipartite subgraph. For instance, this is the case of graphs with no induced path of length  $k$  nor its complement [3]. Then we can deduce the existence of a  $\mathcal{O}(n^{k(c)})$  CS-separator for  $\mathcal{C}$  [2]. Moreover, if  $\mathcal{C}$  is defined as the class of  $H$ -free graphs for some graph  $H$ , then we can also deduce that the Erdős-Hajnal property is verified [1, 10]: there exists a constant  $\delta(H)$  such that in every graph  $G \in \mathcal{C}$ , there exists a clique or a stable set of size  $|V(G)|^{\delta(H)}$ . It has been proved that there exist Berge graphs that do not have a  $c.n$  strongly complete or anticomplete pseudo-bipartite subgraph for any  $c > 0$  [9], thus Theorem 6.1 shows that Berge graphs with no balanced skew partition have some specific structure that does not exist in all Berge graphs. Unfortunately, we cannot deduce the existence of a polynomial CS-separator from 6.1, because our class is not closed under taking induced subgraphs: deleting a vertex might create balanced skew partitions.

In the following, any trigraph  $T$  is given a weight function  $w : V(T) \cup \sigma(T) \rightarrow \mathbb{N}$  and we require that  $w(v) > 0$  for  $v \in V(T)$  and that every switchable pair of non-zero weight must be labeled “2-join” or “complement 2-join”. For every subset  $V' \subseteq V(T)$ , we denote  $w(V') = \sum_{v \in V'} w(v) + \sum_{u, v \in V'} w(uv)$ . In this context, we redefine the size  $n$  of a trigraph  $T$  to be  $w(V(T))$ , and given a constant  $c < 1$ , we say that  $T$  has a  $c.n$  strongly complete or anticomplete pseudo-bipartite subgraph if there exists  $V_1, V_2 \subseteq V(T)$  such that  $w(V_1), w(V_2) \geq c.n$  and  $V_1$  is strongly complete or strongly anticomplete to  $V_2$ . Observe that if  $w$  is the weight function such that  $w(v) = 1$  for every  $v \in V(T)$  and  $w(uv) = 0$  for every  $uv \in \sigma(T)$  we obtain the above notion.

The idea is to contract vertices of  $T$ , while preserving the corresponding weight, until getting a basic one. Let  $T$  be a trigraph with a weight function  $w$ ,  $(X_1, X_2)$  be a proper 2-join in  $T$  or in the complement of  $T$  such that  $w(X_1) \geq w(X_2)$ , and  $(A_1, B_1, C_1, A_2, B_2, C_2)$  a split of  $(X_1, X_2)$ . Let us define the trigraph  $T'$  with weight function  $w'$  to be the contraction of  $T$ , denoted  $(T, w) \rightsquigarrow (T', w')$ .

$T'$  is the block of decomposition  $T_{X_1}$  and its weight function  $w'$  is defined as follow (see Figure2):

- On the vertices of  $X_1$ , we define  $w' = w$ .
- On the marker vertices  $a, b$ , we define  $w'(a) = w(A_2)$  and  $w'(b) = w(B_2)$
- If the marker vertex  $c$  exists, we define  $w'(c) = w(C_2)$ . Otherwise we define  $w'(ab) = w(C_2)$  and we label  $ab$  according to the type of  $(X_1, X_2)$ .

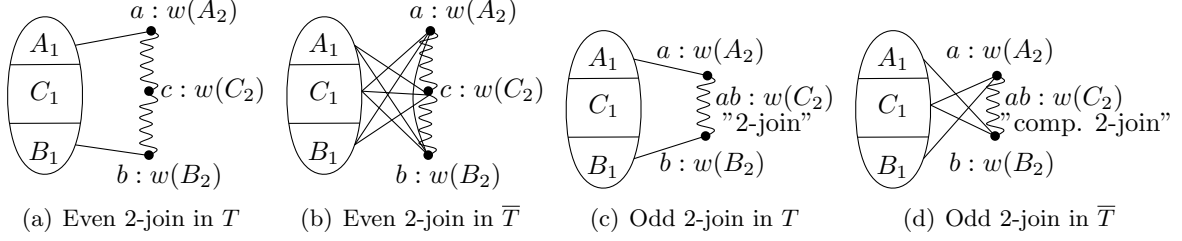


Figure 2: Description of the contraction of  $T$  if  $(A_1, B_1, C_1, A_2, B_2, C_2)$  is a split of a 2-join in  $T$  or in the complement  $\bar{T}$ . A line between a vertex  $x \in \{a, b, c\}$  and a set  $V \in \{A_1, B_1, C_1\}$  means that  $x$  is strongly complete to  $V$ , otherwise  $x$  is strongly anticomplete to  $V$ .

We define  $a$  (resp.  $b, v \in X_1$ ) to be the *representant* of  $A_2$  (resp.  $B_2, v$ ) and for every vertex  $v \in A_2$  (resp.  $B_2, X_1$ ) we note  $v \rightarrow a$  (resp.  $v \rightarrow b, v \rightarrow v$ ). Depending on the existence of  $c$ ,  $c$  or  $ab$  is the *representant* of  $C_2$  and for every vertex  $v \in C_2$  we note  $v \rightarrow c$  or  $v \rightarrow ab$ .

If  $(T, w) \rightsquigarrow (T', w')$  and  $V' \subseteq V(T')$ , we also denote  $V \rightarrow V'$  if  $V = \{v \in V(T) \mid \exists v' \in V' v \rightarrow v' \vee \exists u' v' \in V'^2 v \rightarrow u' v'\}$ . We denote by  $\rightarrow^*$  (resp.  $\rightsquigarrow^*$ ) the transitive closure of  $\rightarrow$  (resp.  $\rightsquigarrow$ ).

**Lemma 6.2.** *If  $T$  is a trigraph with weight function  $w$  and  $(T, w) \rightsquigarrow^* (T', w')$  then*

- $w'(V(T')) = w(V(T))$
- *If  $T'$  has a  $c.n$  strongly complete or anticomplete pseudo-bipartite subgraph  $(V_1, V_2)$ , then  $T$  also has.*

*Proof.* The first item is straightforward. Suppose  $T'$  has a  $c.n$  strongly complete or anticomplete pseudo-bipartite subgraph  $(V_1, V_2)$ . Then there exists  $W_1, W_2 \subseteq V(T)$  such that  $W_1 \rightarrow^* V_1$  and  $W_2 \rightarrow^* V_2$ . Since the contraction does not create strongly adjacency and strongly antiadjacency that did not previously exist, if  $V_1, V_2$  are strongly complete (resp. anticomplete),  $W_1, W_2$  also are. Moreover,  $w(W_1) = w'(V_1)$  and  $w(W_2) = w'(V_2)$ . So  $(W_1, W_2)$  is a  $c.n$  strongly complete or anticomplete pseudo-bipartite subgraph of  $T'$ .  $\square$

**Lemma 6.3.** *Let  $0 < c < 1/6$ . Let  $(T, w)$  be a weighted trigraph of  $\mathcal{F}$  with no balanced skew partition such that  $w(x) < c.n$  for every  $x \in V(T) \cup \sigma(T)$ . Either  $T$  has  $c.n$  strongly complete or anticomplete pseudo-bipartite subgraph, or there exists a basic trigraph  $T'$  with weight  $w'$  such that  $(T, w) \rightsquigarrow^* (T', w')$  and for every  $x \in V(T') \cup \sigma(T')$ ,  $w(x) < c.n$ .*

*Proof.* We prove the result by induction on  $T$ , using the decomposition result on trigraphs of  $\mathcal{F}$  (Theorem 3.4). If  $T$  is not basic, then it admits a proper 2-join or a proper 2-join in the complement. The problem being autocomplementary, we only show here the case of a 2-join  $(X_1, X_2)$  in  $T$ . By symmetry, suppose  $w(X_1) \geq w(X_2)$  and thus  $w(X_1) \geq n/2$ . Let  $(A_1, B_1, C_1, A_2, B_2, C_2)$  be a split of  $(X_1, X_2)$ . By property of  $X_1$ ,  $\max(w(A_1), w(B_1), w(C_1)) \geq n/6 \geq c.n$ . Thus if  $\max(w(A_2), w(B_2), w(C_2)) \geq c.n$ , there is a  $c.n$  pseudo-bipartite strongly complete or anticomplete subgraph. Otherwise,  $(T, w) \rightsquigarrow (T', w')$  with  $w'(x) < c.n$  for every  $x \in V(T') \cup \sigma(T')$  and  $T' \in \mathcal{F}$  with no balanced skew partition by Theorem 3.5. So we can apply the induction hypothesis and either find a basic trigraph  $T''$  such that  $(T, w) \rightsquigarrow (T', w') \rightsquigarrow^* (T'', w'')$  and  $w''(x) < c.n$  for every  $x \in V(T'') \cup \sigma(T'')$ ; or find a  $c.n$  complete or anticomplete pseudo-bipartite subgraph in  $T'$ , and thus in  $T$  by Lemma 6.2.  $\square$

If  $T$  is a basic trigraph with weight function  $w$ , we want to transform  $T$  into a vertex-only weighted trigraph, by transferring the weight on the switchable pairs to new vertices. We define the *extension*  $T'$  of  $T$  as the trigraph with weight function  $w' : V(T') \rightarrow \mathbb{N} \setminus \{0\}$  defined in the following way:  $V(T') = V(T) \cup \{v_{ab} | ab \in \sigma(T), w(ab) > 0\}$ ,  $w'(v) = w(v)$  for  $v \in V(T)$ , the label of  $ab$  is given to  $v_{ab}$ ,  $w'(v_{ab}) = w(ab)$ ,  $\theta(av_{ab}) = \theta(bv_{ab}) = 0$ , and if  $u \in V \setminus \{a, b\}$ , then  $\theta(uv_{ab}) = -1$  if the label of  $ab$  is "2-join" and  $\theta(uv_{ab}) = 1$  if the label is "complement 2-join". Observe that  $ab$  was the representant of  $C_2$  of the split  $(A_1, B_1, C_1, A_2, B_2, C_2)$  of the 2-join, and  $v_{ab}$  takes the place back as a contraction of  $C_2$ , since it has same weight and same strong adjacency and strong antiadjacency with the rest of the graph. It was not possible to keep a contraction of  $C_2$  at each step and simultaneous stay in  $\mathcal{F}$  and not create a balanced skew partition, which is a key ingredient of Lemma 6.3. Observe finally that the newly added vertices labeled by "2-join" form a stable set and those labeled by "complement 2-join" form a clique.

**Lemma 6.4.** *Let  $(T, w)$  be a weighted trigraph of  $\mathcal{F}$  with no balanced skew partition such that  $w(uv) = 0$  for every  $uv \in \sigma(T)$ . Suppose  $(T, w) \rightsquigarrow^* (T', w')$  and let  $T''$  with weight  $w''$  be the extension of  $T'$ . If  $T''$  admits a  $c.n$  strongly complete or anticomplete pseudo-bipartite subgraph, then  $T$  also does.*

*Proof.* Let  $(V_1, V_2)$  be a  $c.n$  strongly complete or anticomplete pseudo-bipartite subgraph in  $T''$ . Let  $X_1 \subseteq V_1$  be the subset of vertices of  $V_1$  with a label. Then  $V_1 \setminus X_1 \subseteq V(T')$  and there exists  $W_1 \subseteq V(T)$  such that  $W_1 \rightarrow^* V_1 \setminus X_1$ . Let  $Y_1 = \{v \in V(T) | \exists v_{ab} \in X_1, v \rightarrow ab\}$ . Then  $w(W_1 \cup Y_1) = w''(V_1) \geq c.n$ . We define similarly  $X_2, W_2$  and  $Y_2$  and we have the similar inequality  $w(W_2 \cup Y_2) = w''(V_2) \geq c.n$ . Moreover,  $W_1 \cup Y_1$  is strongly complete (resp. anticomplete) to  $W_2 \cup Y_2$  if  $V_1$  is strongly complete (resp. anticomplete) to  $V_2$ . Thus  $T$  has a  $c.n$  strongly complete or anticomplete pseudo-bipartite subgraph.  $\square$

**Lemma 6.5.** *If  $T'$  is the extension of a basic trigraph  $T$  of  $\mathcal{F}$  with no balanced skew partition with weight function  $w_0$  and  $w_0(x) < c.n$  for every  $x \in V(T) \cup \sigma(T)$ , then  $T'$  admits a  $c.n$  pseudo-bipartite complete or anticomplete subgraph if  $c \leq 1/148$ .*

Before going to the proof, we need a technical lemma. A graph  $G$  has  $m$  *multi-edges* if its set of edges  $E$  is a multiset of  $V^2 \setminus \{xx | x \in V(G)\}$  of size  $m$ : there can be several edges between two distinct vertices. An edge  $uv$  has two *extremities*  $u$  and  $v$ . The degree of  $v \in V(G)$  is  $d(v) = |\{e \in E | v \text{ is an extremity of } e\}|$ .

**Lemma 6.6.** *Let  $G$  be a bipartite graph  $(A, B)$  with  $m$  multi-edges and maximum degree  $c.m$  with  $c < 1/3$ . Then there exist two subsets  $E_1, E_2$  of edges of  $G$  such that  $|E_1|, |E_2| \geq m/48$  and if  $e_1 \in E_1, e_2 \in E_2$  then  $e_1$  and  $e_2$  do not have a common extremity.*

*Proof.* If  $m \leq 48$ , it is enough to find two edges with no common extremity. Such two edges always exists since the maximum degree is bounded by  $c.m$  so no vertex can be a common extremity to every edge. Otherwise, assume  $m > 48$  and let us consider a random uniform partition  $(U, U')$  of the vertices. For every pair of distinct edges  $e_1, e_2$ , consider the random variable  $X_{e_1, e_2} = 1$  if  $(e_1, e_2) \in (U^2 \times U'^2) \cup (U'^2 \times U^2)$ , and 0 otherwise. If  $e_1$  and  $e_2$  have at least one common extremity, then  $\Pr(X_{e_1, e_2} = 1) = 0$ , otherwise  $\Pr(X_{e_1, e_2} = 1) = 1/8$ . We define the following:

$$\begin{aligned} p &= \{(e_1, e_2) \in E^2 | e_1 \text{ and } e_2 \text{ do not have a common extremity}\} \\ p_A &= \{(e_1, e_2) \in E^2 | e_1 \text{ and } e_2 \text{ do not have a common extremity in } A\} \\ q_A &= \{(e_1, e_2) \in E^2 | e_1 \neq e_2 \text{ and } e_1 \text{ and } e_2 \text{ have a common extremity in } A\} \end{aligned}$$

We define similarly  $p_B$  and  $q_B$ . Assume that  $p \geq \frac{1}{3} \binom{m}{2}$ . Then

$$\mathbb{E}\left(\sum_{\substack{e_1, e_2 \in E \\ e_1 \neq e_2}} X_{e_1, e_2}\right) = \sum_{\substack{e_1, e_2 \in E \\ e_1 \neq e_2}} \Pr(X_{e_1, e_2} = 1) = \frac{p}{8} \geq \frac{1}{24} \binom{m}{2}.$$

Thus there exists a partition  $(U, U')$  such that

$$\sum_{\substack{e_1, e_2 \in E \\ e_1 \neq e_2}} X_{e_1, e_2} \geq \frac{1}{24} \binom{m}{2}.$$

Let  $E_1 = E \cap U^2$  and  $E_2 = E \cap U'^2$ . Then  $|E_1|, |E_2| \geq m/48$ , otherwise

$$\sum_{\substack{e_1, e_2 \in E \\ e_1 \neq e_2}} X_{e_1, e_2} = |E_1| \cdot |E_2| < \frac{m}{48} \cdot \left(1 - \frac{1}{48}\right) m \leq \frac{1}{24} \binom{m}{2},$$

a contradiction. So  $E_1$  and  $E_2$  satisfy the requirements of the lemma. We finally have to prove that  $p \geq \frac{1}{3} \binom{m}{2}$ . The intermediate key result is that  $p_A \geq 2q_A$ . Number the vertices of  $A$  from 1 to  $|A|$  and recall that  $d(i)$  is the degree of  $i$ . Then  $\sum_{i=1}^{|A|} d(i) = m$  and

$$p_A = 1/2 \left( \sum_{i=1}^{|A|} d(i)(m - d(i)) \right) = 1/2 \left( \left( \sum_{i=1}^{|A|} d(i) \right)^2 - \sum_{i=1}^{|A|} (d(i))^2 \right) = 1/2 \left( \sum_{\substack{i, j=1 \\ i \neq j}}^{|A|} d(i)d(j) \right)$$

$$q_A = 1/2 \left( \sum_{i=1}^{|A|} d(i)(d(i) - 1) \right) = 1/2 \left( \sum_{i=1}^{|A|} (d(i))^2 - m \right)$$

Consequently,

$$2p_A - (4q_A + 2m) = \sum_{i=1}^{|A|} d(i) \left( \sum_{\substack{j=1 \\ j \neq i}}^{|A|} d(j) - 2d(i) \right) = \sum_{i=1}^{|A|} d(i) \left( \sum_{j=1}^{|A|} d(j) - 3d(i) \right) = \sum_{i=1}^{|A|} d(i) (m - 3d(i))$$

But for every  $i$ ,  $d(i) \leq c.m \leq m/3$  thus  $m - 3d(i) \geq 0$ . Consequently,  $2p_A - (4q_A + 2m) \geq 0$  and thus  $p_A \geq 2q_A$ . But  $p_A + q_A = \binom{m}{2}$  so  $q_A \leq \frac{1}{3} \binom{m}{2}$ . Similarly,  $p_B \geq 2q_B$  and  $q_B \leq \frac{1}{3} \binom{m}{2}$ . Finally,

$$p \geq \binom{m}{2} - q_A - q_B \geq \binom{m}{2} - \frac{2}{3} \binom{m}{2} \geq \frac{1}{3} \binom{m}{2}.$$

□

*Proof of Lemma 6.5.* Let  $w$  be the weight function on  $T'$ . Since the problem is autocomplementary, it suffices to prove it if  $T$  is a bipartite trigraph, a doubled trigraph or a line trigraph. If  $T$  is a doubled trigraph, then  $T$  has a good partition  $(X, Y)$ . In fact,  $X$  is the union of two stable sets  $X_1$  and  $X_2$  and  $Y$  is the union of two cliques  $Y_1$  and  $Y_2$ . Thus  $T'$  is the union of three stable sets

$X_1, X_2, X_3$  ( $X_3$  is the set of vertices labeled "2-join") and three cliques  $Y_1, Y_2, Y_3$  ( $Y_3$  is the set of vertices labeled "complement 2-join"). There exists one set  $Z$  among these six sets of weight at least  $n/6$ . Since every vertex of  $Z$  has weight at most  $c.n$ , we can split  $Z$  into  $(Z_1, Z_2)$  with  $w(Z_1), w(Z_2) \geq n/12 - c.n \geq c.n$  and  $Z_1$  is either strongly complete (if  $Z$  is a clique) or strongly anticomplete (if  $Z$  is a stable set) to  $Z_2$ . The same argument apply if  $T$  is a bipartite trigraph, which is the union of two stable sets.

It becomes more complicated if  $T$  is a line trigraph. Let  $X$  be the stable set of vertices labeled "2-join" in  $T'$ ,  $Y$  be the clique of vertices labeled "complement 2-join", and  $Z = V(T') \setminus (X \cup Y)$ . By definition of  $T$  being a line trigraph, the full realization of  $T'|Z$  is the line graph of a bipartite graph  $G$ , and every clique of  $T'|Z$  of size at least three is a strong clique. If  $v_{ab} \in X$ , then the full realization of  $T'|(Z \cup \{v_{ab}\})$  also is the line graph of a bipartite: indeed,  $v_{ab}$  is semiadjacent to exactly  $a$  and  $b$ , and antiadjacent to the rest of the vertices. By assumption on the cliques of size three of  $T$ , there cannot be a vertex  $d \in Z$  adjacent to both  $a$  and  $b$ . This means that the common extremity  $x$  of  $a$  and  $b$  in  $G$  has degree exactly two. Add the edge  $v_{ab}$  between  $x$  and a newly added vertex, then the full realization of  $T'|(Z \cup \{v_{ab}\})$  is the line graph of a bipartite graph. By iteration, the full realization  $T'|(Z \cup X)$  also is.

Now we distinguish two cases: if there exists a clique  $K$  of weight  $w(K) \geq 4c.n$  in  $T'$ , then we can split  $K$  into  $(K_1, K_2)$  with  $w(K_1), w(K_2) \geq 4c.n/2 - c.n \geq c.n$  and  $K_1$  is strongly complete to  $K_2$ . Otherwise, observe that in  $Z \cup X$ , every switchable component has at most three vertices. Select the one in each component with the largest weight (in case of ties, we cut them arbitrarily) to get a set of vertices  $V' \subseteq Z \cup X$  inducing no switchable pair, i.e.  $T'|V'$  is a graph.  $T'|V'$  is a subgraph of the full realization of  $T'|(Z \cup X)$  so it is the line graph of a bipartite graph  $G$ . Instead of keeping positive integer weight on the edges of  $G$ , we transform each edge  $xy$  of weight  $m$  into  $m$  edges  $xy$ . The inequality  $w(K) \leq 4c.n$  for every clique  $K$  implies on one hand that the maximum degree of a vertex of  $G$  is at most  $4c.n$ , and on the other hand that  $n' = w(V') \geq (n - w(Y))/3 \geq n(1 - 4c)/3$  since  $Y$  is a clique. Lemma 6.6 prove the existence of two subsets  $E_1, E_2$  of edges of  $G$  such that  $|E_1|, |E_2| \geq n'/48$  and if  $e_1 \in E_1, e_2 \in E_2$  then  $e_1$  and  $e_2$  do not have a common extremity. This corresponds in  $T'$  to a  $n(1 - 4c)/144 \geq c.n$  strongly anticomplete pseudo-bipartite subgraph.  $\square$

*Proof of Theorem 6.1.* Let  $c = 1/148$ . If  $n < 1/c$ , there always exists a strong edge or a strong anti-edge  $uv$  by definition of  $\mathcal{F}$ , and we define  $V_1 = \{u\}$  and  $V_2 = \{v\}$ . Otherwise, give to  $T$  the weight function  $w$  such that  $w(v) = 1$  for every  $v \in V(T)$  (in particular,  $w(V(T)) = n$ ). Apply Lemma 6.3 to  $T$  to either get a  $c.n$  complete or anticomplete pseudo-bipartite subgraph, or to contract  $T$ : there exists a basic trigraph  $T'$  such that  $(T, w) \rightsquigarrow^* (T', w')$ . Apply then Lemma 6.5 to find a  $c.n$  complete or anticomplete pseudo-bipartite subgraph in the extension of  $T'$ , and conclude of the existence of such a subgraph thanks to Lemma 6.4.  $\square$

## 6.2 On the closure $\overline{\mathcal{C}^{\leq k}}$ of $\mathcal{C}$ by generalized $k$ -join

In fact, the method of contraction of a 2-join used in the previous subsection can easily be adapted to a generalized  $k$ -join. We only require that the basic class  $\mathcal{C}$  of graphs is closed under induced subgraphs to make it work on  $\mathcal{C}^{\leq k}$ . We invite the reader to refer to Section 5 for the definitions of a generalized  $k$ -join and the classes  $\mathcal{C}^{\leq k}$  and  $\overline{\mathcal{C}^{\leq k}}$ . We obtain the following result:

**Theorem 6.7.** *Let  $k \in \mathbb{N} \setminus \{0\}$ ,  $0 < c < 1/2$  and  $\mathcal{C}$  be a hereditary class of graphs such that for every  $G \in \mathcal{C}$  and for every weight function  $w : V(G) \rightarrow \mathbb{N} \setminus \{0\}$  such that  $w(v) < c.n$  for every  $v \in V(G)$ ,  $G$  has a  $c.n$  strongly complete or anticomplete pseudo-bipartite subgraph. Then*

every trigraph  $T$  of  $\overline{\mathcal{C}^{\leq k}}$ , with at least  $k/c$  vertices, has a  $(c/k).n$  strongly complete or anticomplete pseudo-bipartite subgraph.

To prove this theorem, we define the contraction of a generalized  $k$ -join. It is simpler than for the 2-join because no weight and no label is ever given to a switchable pair: let  $T$  be a trigraph with weight function  $w : V(T) \rightarrow \mathbb{N} \setminus \{0\}$ , and assume  $T$  is the generalized  $k$ -join of  $T_1$  and  $T_2$ . We follow the notations introduced in the definition of the generalized  $k$ -join, in particular  $V(T)$  is partitioned into  $(A_1, \dots, A_r, B_1, \dots, B_s)$ . Free to exchange  $T_1$  and  $T_2$ , assume  $w(\cup_{j=1}^r A_j) \geq w(\cup_{i=1}^s B_i)$ . Then the contraction of  $T$  is the trigraph  $T' = T_1$  with weight  $w'$  defined by  $w'(v) = w(v)$  if  $v \in \cup_{j=1}^r A_j$ , and  $w'(b_i) = w(B_i)$  for  $1 \leq i \leq s$ . We still denote the contraction operation by  $(T, w) \rightsquigarrow (T', w')$ .

Observe that Lemma 6.2 still holds in this context, and we obtain the following lemma:

**Lemma 6.8.** *Let  $0 < c < 1/(2k)$ . Let  $(T, w)$  be a weighted trigraph of  $\overline{\mathcal{C}^{\leq k}}$  such that  $w(v) < c.n$  for every  $v \in V(T)$ . Either  $T$  has  $c.n$  strongly complete or anticomplete pseudo-bipartite subgraph, or there exists a trigraph  $T' \in \mathcal{C}^{\leq k}$  with weight  $w'$  such that  $(T, w) \rightsquigarrow^* (T', w')$  and for every  $v \in V(T')$ ,  $w(v) < c.n$ .*

*Proof.* Similar to Lemma 6.3 □

*Proof of Theorem 6.7.* Let  $T$  be a trigraph of  $\overline{\mathcal{C}^{\leq k}}$ . We define the weight  $w(v) = 1$  for every  $v \in V(T)$ . Apply Lemma 6.8 to get either a  $(c/k).n$  strongly complete or anticomplete pseudo-bipartite subgraph, or to get a trigraph  $T' \in \mathcal{C}^{\leq k}$  such that  $(T, w) \rightsquigarrow^* (T', w')$  and  $w'(v) < (c/k).n$  for every  $v \in V(T')$ . For every switchable component of  $T'$ , select the vertex with the biggest weight and delete the others. We obtain a graph  $G \in \mathcal{C}$  and define  $w''(v) = w'(v)$  on its vertices. Observe that  $w''(V(G)) \geq w'(V(T'))/k$  since every switchable component has size  $\leq k$ , and that for every  $v \in V(G)$ ,  $w''(v) < (c/k).w'(V(T')) \leq c.w''(V(G))$ . Then there exists  $V_1, V_2 \subseteq V(G)$  such that  $w''(V_1), w''(V_2) \geq c.w''(V(G))$  and  $V_1$  is either strongly complete or strongly anticomplete to  $V_2$ . Then  $w'(V_1), w'(V_2) \geq (c/k).w'(V(T'))$  and  $V_1$  is either strongly complete or strongly anticomplete to  $V_2$  in  $T'$ . Thus  $T'$  has a  $(c/k).n$  strongly complete or anticomplete pseudo-bipartite subgraph. We conclude to the existence of such a subgraph in  $T$  thanks to Lemma 6.2. □

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